Notes on Microeconomics
Game Theory

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Chapter 1

Single person problem: choice under uncertainty

Game Theory is about strategic interactions. It is formal since it uses mathematical models and strategic, since what each player obtains also depends on what other players do. In particular, each player faces uncertainty about what other players do.

Therefore, the founding block of Game Theory is the analysis of the Single Person Decision Theory, i.e., the analysis of choice under certainty.

First, we denote the fundamentals of the problem:

The set of actions is:

$$A = \{a_1, a_2, \ldots, a_n\}$$

The set of states is:

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$$

The *payoff function* is:

$$u : A \times \Omega \longrightarrow \mathbb{R}$$

This function yields a utility denoted as $u(a, \omega_j)$. If:

$$u(a_i, \omega_j) > u(a_k, \omega_j)$$

It means that in the state j, the decision maker strictly prefers action i to action k.

Let's see a very simple example: one has to decide if to take an umbrella when leaving the house. She faces two possible actions, $A = \{\text{Taking an umbrella}, \text{Not taking an umbrella}\}$ and faces two possible states of the world, $\Omega = \{\text{Sunny}, \text{Rainy}\}$. We could represent this situation in the following simple matrix:

	Sunny	Rainy
No Umbrella	3	0
Umbrella	0	3

Then the combination $\{\text{No Umbrella}, \text{Sunny}\}$ gives a total payoff of 3, $\{\text{No Umbrella}, \text{Rainy}\}$ gives 0, and so on.

1.1 Domination and Expected Utility

Let's now introduce the concept of domination.

Definition 1.1.1 (Domination)**.** Action a strictly dominates action b if $u(a,\omega) > u(b,\omega)\ \ \forall\omega \in \Omega$. Then b is strictly dominated

Definition 1.1.2 (Weak Domination)**.** Action a weakly dominates b if $u(a,\omega) \geq u(b,\omega)\ \ \forall\omega \in \Omega$ and $u(a,\omega) > u(b,\omega)$ for some $\omega \in \Omega$.

Examples:

In the following matrix, a strictly dominates b. Indeed for each ω_i the payoffs of a are greater than those of b.

	ω_1	ω_2	ω_3
a	1	3	0
b	0	-1	5

Instead, in the matrix below a only weakly dominates b. Indeed, for ω_3, the payoffs are equal.

	ω_1	ω_2	ω_3
a	1	3	0
b	0	-1	0

Therefore, we say that a is strictly (weakly) dominant if it strictly (weakly) dominates any $a' \neq a$. This roughly means that a is optimal no matter what. Then, unless we have dominant strategies, we must think about probabilities (or beliefs).

Put in a simple way, beliefs are probability distributions about different states of the world. They can be represented as the vector q:

$$q = (q(\omega_1), q(\omega_2), \ldots, q(\omega_m))$$

where each $q(\omega_j)$ represents the probability of an event. So $q(\omega_j) \geq 0 \quad j = 1, \ldots, m$ and $\sum_{j=1}^{m} q(\omega_j) = 1$. Finally, we also define $\Delta(\Omega)$ as the set of all possible beliefs.

Furthermore, agents are expected utility maximizers (von Neumann-Morgenstern Utility Maximizers). This means that we can write their utility function as follows:

$$U(a,q) = \sum_{j=1}^{n} q(\omega_j)u(a,\omega_j)$$

Where U is a real-valued utility function defined as: $U : A \times \Delta(\Omega) \longrightarrow \mathbb{R}$. Then, for two actions a, b we say that $u(a,\omega) > u(b,\omega)$ if:

$$\sum_{j=1}^{m} q(\omega_j)u(a,\omega_j) > \sum_{j=1}^{m} q(\omega_j)u(b,\omega_j)$$

1.2 Best Response and Never a Weak Best Response

Two other fundamental concepts must be defined: Best Response (BR) and Never a Weak Best Response (NWBR).

Definition 1.2.1 (Best Response)**.** Given (A, Ω, u, q) action a is a Best Response to $q \in \Delta(\Omega)$ if and only if $U(a, q) \geq U(a', q) \quad \forall a' \in A$.

Let's see an example. Recall the situation above, the decision between taking an umbrella or not, facing the possible states of the world {Sunny, Rainy}. See the matrix below:

	Sunny	Rainy
No Umbrella	5	0
Umbrella	1	3

Now one can attribute a probability to the two different states, namely $q(q_1, q_2) \equiv q = (q_1, (1 - q_1))$. Then we can compute the utility associated with each action:

$$U(\text{NU}, q) = 5q_1 + (1 - q_1)0 = 5q_1$$

and

$$U(\text{U}, q) = q_1 + 3(1 - q_1) = 3 - 2q_1$$

To see what is optimal, let's see Figure 1. Then before $q_1 = \frac{3}{7}$ taking an umbrella is the optimal action. After $q_1 = \frac{3}{7}$ Not taking an umbrella is optimal. Finally, when $q_1 = \frac{3}{7}$, the optimal choice is to be indifferent.

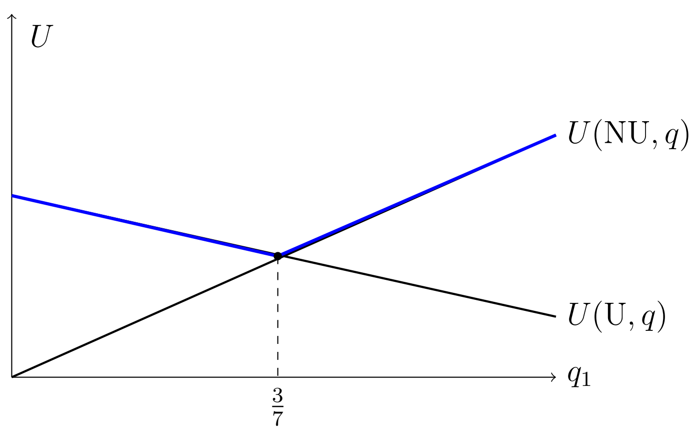


Figure 1.1: Best Response

Formally, we can write $BR(q)$ as:

$$BR(q) = \begin{cases} U & \text{if } q_1 < \frac{3}{7} \\ U \sim NU & \text{if } q_1 = \frac{3}{7} \\ NU & \text{if } q_1 > \frac{3}{7} \end{cases}$$

Let's now relate the Best Response and Domination.

Proposition 1. *If a is (weakly) dominant, then a is a Best Response to any belief $q \in \Delta(\Omega)$.*

Proof. $\forall a' \neq a$ we can write $u(a,\omega) \geq u(a',\omega) \;\; \forall \omega$, with at least one strict inequality. In terms of expected utility, given a vector of beliefs $q = (q_1, \ldots, q_m)$, we can write:

$$U(a,q) = \sum_{j=1}^{m} q(\omega_j)u(a,\omega_j) \geq \sum_{j=1}^{m} q(\omega_j)u(a'\omega) = U(a',q)$$

Since $u(a,\omega_j) \geq u(a',\omega_j)$. □

Another fundamental idea is that of *Never a Weak Best Response*. Plainly speaking, an action a is NWBR when, no matter the belief, it is never optimal.

Definition 1.2.2 (NBWR)**.** Action a is Never a Weak Best Response if it does not exist a $q \in \Delta(\Omega))$ such that a is $BR(q)$.

From this definition, a fundamental proposition can be stated.

Proposition 2. *If a is strictly dominated, then it is an NWBR.*

Proof. To see this just apply the definition. a is strictly dominated if exist a a' such that $u(a',\omega) > u(a,\omega), \quad \forall \omega \in Omega$. Then take a $q \in \Delta(\Omega)$. We can write:

$$U(a',q) = \sum_{j=1}^{m} q(\omega_j)u(a',\omega_j) > \sum_{j=1}^{m} q(\omega_j)u(a,\omega_j) = U(a,q)$$

□

Note that the result above holds only with strict domination. With weak domination, the proposition is not true. To see this, let's look at a simple counterexample:

	ω_1	ω_2
a	2	3
b	2	5

a is weakly dominated by b. But if we allow for $q = (1,0)$, then you can choose both a and b, since $U(a,q) = 1(2) + 0(3) = 1(2) + 0(5) = U(b,q)$.

Proposition 2 states that if an action is strictly dominated, then it is never a Weak Best Response. Then, one can ask if the converse is also true, namely that if an action is Never a Weak Best Response, it is strictly dominated. The answer is yes, but to see this one has to introduce the notion of *Mixed Strategies*.

1.3 Mixed Strategies

Let's start with a simple example.

	ω_1	ω_2
a	3	0
b	0	3
c	1	1

It is apparent that c is not strictly dominated by any other action. Take $q = (q, 1-q)$ and let's compute the following:

$$U(a,q) = 3q \geq 1.5 \quad \text{for } q \geq \frac{1}{2}$$

$$U(b,q) = 3 - 3q \geq 1.5 \quad \text{for } q \leq \frac{1}{1}$$

$$U(c,q) = 1$$

Then, no matter what values q takes, you can always guarantee a payoff greater or equal to 1.5. c is an NWBR even if it is only weakly dominated. Let's now introduce an action d as a randomization of a and b.

$$u(d,\omega_1) = \frac{1}{2}u(a,\omega_1) + \frac{1}{2}u(b,\omega_1) = \frac{3}{2}$$

$$u(d,\omega_2) = \frac{1}{2}u(a,\omega_2) + \frac{1}{2}u(b,\omega_2) = \frac{3}{2}$$

So d strictly dominates c. Therefore, when we just look at actions, then it is not true that NWBR implies strict domination. But introducing mixed strategies, it is true, instead.

A mixed strategy is a randomization over actions:

$$\sigma = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_n))$$

Since σ is a probability distribution, then $\sigma(a_i) \geq 0$ and $\sum_{i=1}^{n} \sigma(a_i) = 1$. Furthermore, we can define $\Delta(A)$ as the *Set of all Mixed Strategies*.

Notice that pure strategies are a special case of mixed strategies. Indeed we can write a pure strategy as $(0, 1, 0, \ldots, 0)$.

Given a mixed strategy σ, a useful concept is that of Support for that mixed strategy.

Definition 1.3.1. The Support of σ, denoted *Supp*(σ), is the set of actions for which the attached probability is non-null, i.e.

$$Supp(\sigma) = \{a : \sigma(a) > 0\}$$

Allowing for mixed strategies, the agent now faces two different types of uncertainty. One,σ is *endogenous*, that is, before knowing what state will occur, the agent must decide which strategy to play. The other,q is *exogenous*, and it concerns the beliefs of each player on the probability each state occurs.

Now we can formally write the utility function as:

$$U : \Delta(A) \times \Delta(\Omega) \longrightarrow \mathbb{R}$$

And:

$$\begin{aligned} U(\sigma, q) &= \sum_{a\in A}\sum_{\omega\in\Omega}\sigma(a)q(\omega)u(a,\omega) \\ &= \sum_{a\in A}\sigma(a)\underbrace{\sum_{\omega\in\Omega}q(\omega)u(a,\omega)}_{U(a,q)} \\ &= \sum_{a\in A}\sigma(a)U(a,q) \end{aligned} \tag{1.1}$$

To get rid of zero probabilities, one can write $\sum_{a\in A}\sigma(a)U(a,q)$ as $\sum_{a\in Supp(\sigma)}\sigma(a)U(a,q)$.

We can redefine domination as follows.

Definition 1.3.2. Action a is strictly dominated if it exists a $\sigma \in \Delta(A)$ such that $u(\sigma,\omega) > u(a,\omega)$

An action cannot be strictly dominated by *Pure Strategies*, but be strictly dominated in *Mixed Strategies*. Returning to the example at the beginning of this section, then we can write the following mixed strategy that strictly dominates c: $\sigma = (\frac{1}{2}, \frac{1}{2}, 0)$.

We can now also redefine the notion of Best Response.

Definition 1.3.3 (Best Response Mixed Strategy)**.** $\sigma \in \Delta(A)$ is a BR to q if $U(\sigma,q) \geq U(a,q) \quad \forall a \in A$

Notice that a Mixed Strategy is optimal if it is better to any Pure Strategy, so it is sufficient to check for those.

Proposition 3. *A Mixed Strategy σ is a BR to q if and only if all the actions in the Support of σ are BR to q*

Proof. Take a σ which is BR(q). Then $U(\sigma,q) \geq U(a,q) \;\; \forall a \in A$ implies that $U(\sigma,q) \leq \max U(a,q)$. However, by optimality of σ, we have also $U(\sigma,q) \geq U(a,q)$ for all a. This implies that $U(\sigma,q) = \max U(a,q)$. And $U(a,q) = U(a',q)$ for all $a, a' \in Supp(\sigma)$. $\square$

Given a system of beliefs, a BR always exists. If a, b are BR, any randomization is a BR. Therefore, there are infinitely many. If there is only one BR, this is pure.

Notice also that an action may be strictly dominated by a mixed strategy without being strictly dominated by any pure strategy (see the example at the beginning of this section).

Finally, an important result can be stated concerning mixed strategies and NWBR.

Proposition 4. *A strategy a is NWBR if and only if it is strictly dominated.*

Proof. $\Leftarrow$ If a is strictly dominated, then it is NWBR. This is shown in Proposition 2 above.

Let's see $\Rightarrow$. If a strategy a is NWBR, then it is strictly dominated. Assume, by contradiction, that a is not strictly dominated. Then it can be found some $q \in \Delta(\Omega)$ such that $a \in BR(q)$.

We can write x^a as the vectors of payoffs of action a through different states of the world, namely:

$$x^a = (u(a, \omega_1), u(a, \omega_2), \ldots, u(a, \omega_m)) \in \mathbb{R}^m.$$

And we define as X the set of all possible payoff vectors:

$$X = \{x \in \mathbb{R}^m : x = (u(\sigma, \omega_1), u(\sigma, \omega_2), \ldots, u(\sigma, \omega_m)) \text{ for some } \sigma \in \Delta(A)\}$$

Notice that $x^a \in X$ and X is convex. This means that for any two $x, x' \in X$, also their convex combination $\lambda x + (1 - \lambda)x' \in X, \ \forall x \in X$ and $\lambda \in [0, 1]$.

Finally, define:

$$Y = \{y \in \mathbb{R}^m : y_i > x^a, i = 1, \ldots, m\}$$

As the set of all payoffs that are preferred to x^a.

Notice that X and Y are disjoint, i.e. $X \cap Y = \emptyset$. This can be seen by contradiction. Then assume the existence of a $\tilde{x}$ which belings both to X and Y. This means that $\tilde{x} = (U(\sigma, \omega_1), \ldots, u(\sigma, \omega_m)$ and $u(\sigma, \omega_i) > u(a, \omega_i)$ for all $i = 1, \ldots, m$. Hence $u(\sigma, \omega) > u(a, \omega)$, but this contradicts the hypothesis of a being undominated.

Since X, Y are disjoint, we can apply the *Supporting Hyperplane Theorem.* This states that, given two disjoint, convex sets, then exist a vector $q \in \mathbb{R}^m \neq (0, 0, \ldots, 0)$ and a $c \in \mathbb{R}$ such that:

$$q \cdot x \geq c \ \ \forall x \in A$$

and

$$q \cdot y \leq c \ \ \forall y \in B$$

Roughly speaking, this theorem states that given two disjoint, convex sets, I can always draw a hyperplane (i.e., a vectorial subspace $a_1 \times x_1 + a_2 \dot{x}_2 + \cdots + a_m \dot{x}_m = b$) that separate them.

By this theorem, we can write:

$$q \cdot y \geq \underbrace{x^a \cdot q}_{q} \ \ \forall y \in Y$$

and

$$q \cdot x \leq \underbrace{x^a \cdot q}_{q} \ \ \forall x \in X$$

Notice that $q_i \geq 0$. To see this, assume, by contradiction, it is not, so $q_1 < 0$. Take a $y' \in Y$ such that $y' = (u(a,\omega_1) + \Delta, u(a,\omega_2) + 1, \dots, u(a,\omega_m) + 1)$ and $\Delta > 0$, so $y' \in Y$. By SHT, $q_1 \cdot (u(a,\omega_1) + \Delta) + q_2 \cdot (u(a,\omega_2) + 1) + \dots + q_m \cdot (u(a,\omega_m) + 1) \geq q \cdot x^a$. But taking the limit for $\Delta \to \infty$, this gives $-\infty$, contradicting the SHT. Then, $q_i \geq 0$ for all i.

Let's show also that $\sum_{i=1}^m q_i = 1$. Indeed, since all q_i are positive, we can write q as $(\frac{q_1}{\sum_{i=1}^m q_i}, \frac{q_2}{\sum_{i=1}^m q_i}, \dots, \frac{q_m}{\sum_{i=1}^m q_i})$.

Then q is a belief, i.e. $q \in \Delta(\Omega)$. Recall that we have assumed that $a \in BR(q)$. Then, we have:

$$q \cdot x \leq q \cdot x^a \ \ \forall x \in X$$

Take an action $b \in A$ and define $\tilde{x} = (u(b,\omega_1), u(b,\omega_2), \dots, u(b,\omega_m))$. This implies:

$$\tilde{x} \cdot q \leq x^a \cdot x$$

and:

$$\sum_{i=1}^m q_i \cdot u(b,\omega_i) \leq \sum_{i=1}^m q_i \cdot u(a,\omega_i) \Rightarrow U(b,q) \leq U(a,q)$$

This contradicts the claim that $a \in BR(q)$.

$\square$

Chapter 2

Games in Strategic Form with Complete Information

2.1 Fundamentals

These are situations where players move only *once* and *simultaneously*. Simultaneity means that there is *no coordination*. No player has more information on the actions of other players.

A Game can be defined as follows:

$$G = (S_1, \dots, S_n, u_1, \dots, u_n)$$

A game is characterized by a set of actions, one for each player, and a set of utility functions that give a payoff. For each player i we can define a non-empty set of actions (pure strategies): S_i.

The set of all possible strategy profiles (formally the Cartesian product of the set of all individual strategies):

$$\mathbf{S} = \bigtimes_{j=1}^{n} S_j$$

A strategy profile lists the actions taken by all players. Fixing a player i, we can also write:

$$\mathbf{S}_{-i} = \bigtimes_{j \neq i}^{n} S_j$$

So we can write an entire strategic profile as $s = (s_i, s_{-i})$.

The utility function is formally defined as:

$$u : \mathbf{S} \longrightarrow \mathbb{R}$$

or $u_i(s) = u_i(s_1, \dots, s_n)$.

Mixed strategies are randomizations over possible actions.

$$S_i = (s_i^1, s_i^2, \ldots, s_i^n)$$

Is the set of available strategies for player i. A (generic) mixed strategy over that actions can be written as:

$$\sigma_i = (\sigma_i(s_i^1), \sigma_i(s_i^2), \ldots, \sigma_i(s_i^n))$$

Where $\sigma_i(s_i^j) \geq 0$ is the probability that player i attaches to action j (and therefore $\sum_{j=1}^n \sigma_i(s_i^j) = 1$).

We also define $\Delta(S_i)$ as the set of all mixed strategies for player i.

The support of σ_i is the set of all strategies played with positive probabilities:

$$Supp(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$$

As in the case of pure strategies, we can write mixed strategies as:

$$\sigma = (\sigma_i, \sigma_{-i})$$

Example 1: Battle of Sexes I

A couple, Alice and Bob, must choose where to go out on Friday night. They cannot communicate and can choose between two different alternatives, Football or Opera. The worst outcome for both is that where they do not meet. We can represent their payoffs in the following matrix (the first payoff refers to Alice, the second to Bob):[1]

Alice | Bob	Opera	Football
Opera	3,1	0,0
Football	0,0	1,3

Assuming for Alice and Bob the following mixed strategies:

$$\sigma_A = \left(\frac{1}{3}, \frac{2}{3}\right) \text{ and } \sigma_B = \left(\frac{3}{4}, \frac{1}{4}\right)$$

We can compute the total payoff of Alice associated with these strategies:

[1]To read this matrix: if both Alice and Bob meet at the Opera, she receives a utility of 3 and he 1. If they meet at the Football stadium, he receives 3 and she just 1. If she goes to Opera and he to the Football stadium (and vice-versa), both receive a utility of 0.

$$
\begin{aligned}
U_A(\sigma_A, \sigma_B) = \quad & \underbrace{\left(\frac{1}{3} \cdot \frac{3}{4}\right) \cdot 3}_{(\sigma_A(Op) \cdot \sigma_B(Op)) \cdot U_A(Op, Op)} \quad + \\
& \underbrace{\left(\frac{1}{3} \cdot \frac{1}{4}\right) \cdot 0}_{(\sigma_A(Op) \cdot \sigma_B(F)) \cdot U_A(Op, F)} \quad + \\
& \underbrace{\left(\frac{2}{3} \cdot \frac{3}{4}\right) \cdot 0}_{(\sigma_A(F) \cdot \sigma_B(Op)) \cdot U_A(F, Op)} \quad + \\
& \underbrace{\left(\frac{2}{3} \cdot \frac{1}{4}\right) \cdot 1}_{(\sigma_A(F) \cdot \sigma_B(F)) \cdot U_A(F, F)}
\end{aligned}
\tag{2.1}
$$

This can be rearranged as follows:

$$
\begin{aligned}
& \underbrace{\frac{1}{3}\left[\frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 0\right]}_{\sigma_A(Op) \cdot [U_A(Op, \sigma_B)]} + \underbrace{\frac{2}{3}\left[\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1\right]}_{\sigma_A(F) \cdot [U_A(F, \sigma_B)]} = \\
& = \sum_{\sigma_A \in S_A} \sigma_A(a_A) U_A(a_A, \sigma_B)
\end{aligned}
$$

More in general, the utility of player i from a mixed strategy σ can be computed in the following way:

$$
\begin{aligned}
& U_i(\sigma) = U(\sigma_i, \sigma_{-i}) = \\
& \sum_{(s_1, \ldots, s_m)} \left(\prod_{j=1}^{n} \sigma_j(s_j)\right) u_i(s_1, \ldots, s_m) = \\
& \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \left(\sigma_i(s_i) \prod_{j \neq i} \sigma_j(s_j)\right) U_i(s_i, s_{-i}) = \\
& \sum_{s_i \in S_i} \sigma_i(s_i) \underbrace{\sum_{s_{-i} \in S_{-i}} \left(\prod_{j \neq i} \sigma_j(s_j)\right) U_i(s_i, s_{-i})}_{U_i(s_i, \sigma_{-i})} = \\
& \sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) U_i(s_i, \sigma_{-i})
\end{aligned}
$$

2.2 Dominance in Games

As in the single-person problem, we can assess the actions of each player through the idea of dominance. This means checking for all possible action profiles of the opponent.

Definition 2.2.1 (Strict Domination)**.** A strategy s_i is strictly dominated if it exists a $\sigma_i \in \Delta(S_i)$ such that $u_i(\sigma_i, s_{-i}) > u(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$.

Furthermore, a strategy $s_i \in S_i$ is strictly dominant if and only if it strictly dominates any other $s_i' \neq s_i$.

Definition 2.2.2 (Weak Domination)**.** A strategy s_i is weakly dominated if it exists a $\sigma_i \in \Delta(S_i)$ such that $u_i(\sigma_i, s_{-i}) \geq u(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$, with at least one strict inequality.

Example 2: Prisoner's Dilemma

Two men are arrested for a serious crime, but the police do not have enough evidence to arrest both of them. Still, they do not know that. Furthermore, they cannot communicate, and each prisoner does not trust what the other will do. If both stay silent, they will be charged just with a minor felony (1 year of prison each). If one confesses, he will be absolved and the other arrested, taking the maximum punishment (9 years of prison). Finally, if both confess, they will be condemned to 6 years of prison each. This situation can be represented in the following matrix:

Prisoner 1 | Prisoner 2	Silent	Confess
Silent	-1,-1	-9,0
Confess	0,-9	-6,-6

This is an example of a game with a strictly dominant strategy, namely (Confess, Confess). However, this outcome is Pareto-Dominated by another strategy, (Silent, Silent).

Example 3: Second Price Auctions (Vickrey Auctions)

The basic idea of this kind of auction is that every player makes a bid, and the winner, namely the highest bidder, does not pay the price she bid but instead the price bid by the second-highest bidder.

The setup of this model is the following. $S_i = \mathbb{R}_+$ and each i bid $b_i \in S_i$. Every player has an evaluation v_i of the object to be auctioned. The payoff function is following:

$$u_i(b_1, \dots, b_i, \dots, b_n) = \begin{cases} 0 \ \text{ if } b_j > b_i \text{ for some } j \neq i \\ v_i - b_i \ \ b_i > b_j \geq b_k \text{ for all } k \neq i \\ \frac{v_i - b_i}{m+1} \ \text{ if } b_i \geq b_k \text{ for all } k \end{cases}$$

In words, if the bid of i is inferior to the winning bid, i does not gain anything. If b_i is winning, then what i receives is the difference between her evaluation and b_j (the second highest bid). Finally, in the case of ties, i receives the difference between her evaluation and her bid, dividing the number of bidders who bid the same.

We want to show that bidding her own evaluation is weakly dominant.

Assume $\hat{b} = \max_{j \neq i} b_j$. We have three cases.

First case: $\hat{b} < v_i$. The highest bid is lesser than i's evaluation. Then:

- If i bids her own evaluation, then $b_i = v_i$ i wins and the payoff is $v_i - \hat{b} > 0$.
- $b_i > \hat{b}$ i wins and the payoff is $v_i - \hat{b} > 0$.
- $b_i = \hat{b}$ There is a tie, and the payoff of i is at most $\frac{v_i - \hat{b}}{2} < v_i - \hat{b}$ (in the case the tie is only among two players).
- $b_i < \hat{b}$ i loses and the payoff is 0.

Second case: $\hat{b} > v_i$. The highest bid is greater than i's evaluation. Then:

- $b_i = v_i$ i loses and the payoff is 0.
- $b_i > \hat{b}$ i wins and the payoff is $v_i - \hat{b} < 0$.
- $b_i = \hat{b}$. There is a tie, and the payoff is at most $\frac{v_i - \hat{b}}{n} < 0$.
- $b_i < \hat{b}$. i loses and the payoff is 0.

Third case: $\hat{b} = v_i$. The highest bid is exactly equal to i's evaluation. Then:

- $b_i = v_i$. There is a tie, and the payoff is 0.
- $b_i > \hat{b}$. i wins, and the payoff is 0.
- $b_i = \hat{b}$. There is a tie, and the payoff is 0.
- $b_i < \hat{b}$ i loses and the payoff is 0.

In any case, bidding exactly v_i yields a payoff larger or equal to all other payoffs that one can obtain by bidding something else. Therefore $b_i = v_i$ is a weakly dominant strategy.

2.3 Iterative Removal of Dominated Strategies

Some games can be solved by dominant strategies. Indeed, if players are rational, they play these strategies. Still, notice that most games cannot be solved through dominance.

Furthermore, even if no player has strictly dominant strategies, then there can be strictly dominated strategies. An example is given in the following matrix:

Player 1 | Player 2	Left	Mid	Right
Up	1,0	1,2	0,1
Down	0,3	0,1	2,0

If player 2 is rational, he won't play **Right**, since it is strictly dominated by **Mid** $(2 > 1, 1 > 0)$. We can remove **Right** from the set of actions of 2. So we have the following matrix:

Player 1 | Player 2	Left	Mid
Up	1,0	1,2
Down	0,3	0,1

If player 1 knows that 2 is rational, and she is rational too, then will play **Up** since it is strictly dominant over **Down** $(1 > 0, 1 > 0)$. Therefore we have:

Player 1 | Player 2	Left	Mid
Up	1,0	1,2

Player 2 knows that player 1 knows he is rational, then he will play **Mid** since it is dominant. The game is solved since both players' set of actions are now Singleton. The outcome is (**Up**,**Mid**), and the payoff is $(1, 2)$.

This is an example of the working of an algorithm called "Iterated Deletion of Strictly Dominated Strategies" (IDSDM). According to this procedure, a game is solved if, at the end, the set of all actions is reduced to a Singleton, namely, it contains only one element.

Formally, we have a game $G = (S_1, \dots, S_n, u_1 \dots, u_n)$ and define $G = G^0$.At any step $k - 1$(where $k \geq 1$) we have a game:

$$G^k = (S_1^k, \dots, S_n^k, u_1, \dots, u_n)$$

Where $S_i^k = \{s_i \in S_i^{k-1} : s_i \quad \text{is not strictly dominated in } G^{k-1}\}$ In words, the actions that survive, at any step, are those that are not dominated by any other available action, given the survived actions of other players. Such a deletion process can lead to the following outcome:

$$S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$$

Therefore we can solve a game through IRSDS if and only if S_i^∞ is a Singleton.

Although this procedure is rarely applicable because it requires strict domination and does not work with weak domination (see below), still it displays some fashionable properties. One is that if S_i^∞ is a singleton, then it is a Nash Equilibrium (see below). The other is stated in the following result.

Proposition 5. *The set of strategies that survive IRSDS does not depend on the order of deletion.*

Proof. (Missing)

□

Through an example, let's see why the iterative removal of weakly dominated strategies does not work. Indeed, in this case, the order of deletion matters.

Player 1 | Player 2	Left	Mid	Right
Top	50,0	5,5	1,-10
Bottom	50,50	5,0	0,-10

If we start with player 1, **Top** weakly dominates **Bottom**, so player 1 deletes **Bottom**. In that case, for player 2 **Mid** is strictly dominant, so the outcome of the game is (**Top**,**Mid**).

But if we start with player 2, then he eliminates **Right**, which is strictly dominated. In that case, the game has no solution.

A different procedure is the "Iterative removal of NWBR." Recall that an NWBR is a strategy s_i for which it does not exist a mixed strategy profile such that $s_i \in BR(\sigma_i)$. If we iteratively remove the NWBR, we obtain the set of *Rationalizable Strategies.*

Finally, we relate strict domination and NWBR in the case of games.

Proposition 6. *Taken an game $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$. If $s_i \in S_i$ is strictly dominated, then it is NWBR.*

Proof. Suppose $s_i \in S_i$ is strictly dominated. Then it exists a σ_i such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \;\; \forall s_{-i} \in S_{-i}$. This implies that, for any $\sigma_{-i} \in S_{-i}$:

$$
\begin{aligned}
&u_i(\sigma_i, \sigma_{-i}) = \\
&\sum_{s_i \in S_i} \Big(\prod_{j \neq i} \sigma_j(s_j) \Big) u_i(\sigma_i, s_{-i}) > \\
&\sum_{s_i \in S_i} \Big(\prod_{j \neq i} \sigma_j(s_j) \Big) u_i(s_i, s_{-i}) = \\
&u_i(s_i, s_{-i})
\end{aligned}
$$

□

Still, notice that the converse is true only in the case of 2 players. Then, if $n = 2$, then any NWBR is strictly dominated.

But this is not true if $n > 2$. Indeed if s_i is not strictly dominated, then it exists a $q \in \Delta(\Omega)$ such that $\sum_\Omega q(\omega)u(a, \omega) \geq \sum_\Omega q(\omega)u(b, \omega) \;\; \forall b \in A$. Now, let's $\Omega \equiv S_{-i}$. If s_i is not strictly dominated then it exists a $\beta \in \Delta(S_{-i})$ such that $\sum_{S_{-i}} \beta(s_{-i})u(s_i, s_{-i}) \geq \sum_{\S_{-i}} \beta(s_{-i})u(s'_i, s_{-i}) \;\; \forall s'_i$.

However we cannot show that players' actions come from independent randomization. With more than 2 players, the set of rationalizable strategies is smaller than $\bigtimes S_i^\infty$.

2.4 Nash Equilibrium

There are games that are not solvable through dominance. Then we need a stronger notion, namely that of Nash Equilibrium (NE).

To fully assess what NE is, we need first define the idea of "Best Reply Correspondence".

Definition 2.4.1 (Best Reply Correspondence)**.** Fix $\sigma_{-i} = (\sigma_1, \sigma_2, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$. Then, we define $BR(\sigma_{-i}$ as:

$$BR(\sigma_{-i}) = \{s_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(s_i, s_{-i}) \ \ \forall s_i \in S_i\}$$

Notice that this is equivalent to say that $\sigma_i \in BR(\sigma_i)$ if and only if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \ \ \forall \sigma_i' \in \Delta(S_i)$ (where $u_i(\sigma_i', \sigma_{-i}) = \sum_{S_i} \sigma'(s_i) u_i(s_i, \sigma_{-i})$.)

Proposition 7. *If $\sigma_i \in BR(\sigma_i)$ then $Supp(\sigma_i) \subset BR(\sigma_i)$*

Proof. Notice that we can write $u_i(\sigma_i, \sigma_{-i})$ as $\sum_{s_i \in Supp(\sigma_i)} \sigma_i(s_i) u_i(s_i, s_{-i})$. By definition $u_i(\sigma_i, \sigma_{-i}) \leq \max_{s_i \in Supp(\sigma_i)} u_i(s_i, \sigma_{-i})$. But since σ_i is a BR, then $u_i(\sigma_i, \sigma_{-i}) \geq \max_{s_i \in Supp(\sigma_i)} u_i(s_i, \sigma_{-i})$. Then, $u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in Supp(\sigma_i)} u_i(s_i, \sigma_{-i})$. $\square$

Example 4: Battle of Sexes II

Let's return to Battle of Sexes. This game has the following payoff matrix.

Alice | Bob	Opera	Football
Opera	3,1	0,0
Football	0,0	1,3

For each player we can define the following mixed strategies:

$$\sigma_A = (\alpha, 1-\alpha) \ \text{ and } \ \sigma_B = (\beta, 1-\beta)$$

Where $\alpha, \beta \in [0,1]$. Let's compute $U_A(Op, \sigma_B)$ and $U_A(Op, \sigma_B)$:

$$U_A(Op, \sigma_B) = 3 \cdot \beta + 0 \cdot (1-\beta) = 3 \cdot \beta$$
$$U_A(F, \sigma_B) = 0 \cdot \beta + 1 \cdot (1-\beta) = 1-\beta$$

To find the BR of A to σ_B we must find that β for which $U_A(Op, \sigma_B) \geq U_A(F, \sigma_B)$:

$$3 \cdot \beta \geq 1 - \beta =$$
$$\beta \geq \frac{1}{4}$$

Therefore we have:

$$BR_A(\sigma_B) = \begin{cases} \alpha = 1 \ \text{ if } \beta > \frac{1}{4} \\ \alpha \in [0,1] \ \text{ if } \beta = \frac{1}{4} \text{ (A continuum of BR)} \\ \alpha = 0 \ \text{ if } \beta < \frac{1}{4} \end{cases}$$

This means that if $\beta < \frac{1}{4}$, it is always optimal for Alice to choose Opera. If $\beta > \frac{1}{4}$, it is always optimal to choose Football. If $\beta = \frac{1}{4}$ the BR is a continuum.

This can be represented in Figure 2:

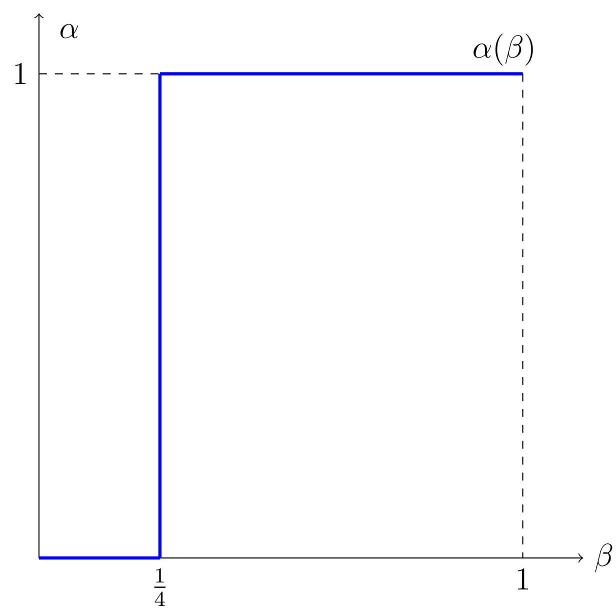


Figure 2.1: Best Reply Correspondence

2.4.1 Pure Strategies Nash Equilibrium

Definition 2.4.2 (Pure Strategies Nash Equilibrium)**.** A strategy profile $s^* = (s_1 \dots, s_n)$ is a Pure Strategy Nash Equilibrium if, $\forall i, \forall s_i \in S_i$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

Or equivalently: s^* is a Pure Strategy Nash Equilibrium if $\forall i$, $s^* \in BR(s_{-i}^*)$.

This means that the NE is a strategy profile where everyone is best responding to others. Simply put, *there are no unilateral profitable deviations.*

If we return to the Prisoner's Dilemma (example 2, see above), for example, the unique NE is (**confess**,**confess**), since any other outcome could provide each player a profitable deviation. If 1 does not confess, 2 does not confess, but then 1 can confess.

In the Battle of Sexes, the Nash Equilibria are the outcomes where both the players stay together, then (**Opera**,**Opera**) or (**Football**,**Football**).

Example 5: Cournot Duopoly (1838)

Two firms simultaneously choose quantities to produce (q_1, q_2) in order to maximize their profits. This is a game theory problem because each firm must take into account also what the other firm will do. Their strategies are $S_1 = S_2 = \mathbb{R}_+$. Both the firms have the following *Inverse Demand Function*:

$$P(q_1 + q_2) = \begin{cases} a - (q_1 + q_2) & \text{if } q_1 + q_2 \leq a \\ 0 & \text{otherwise} \end{cases}$$

Each firm has the following cost function: $C(q_i) = c \cdot q_i$ (marginal costs are constant) and $c > a$ (indeed, if $c > a$, then the costs are greater than any possible a and it is optimal to produce $(0, 0)$).

Finally, the payoffs of each firm are their profits:

$$\pi_1(q_1, q_2) = P(q_1 + q_2) \cdot q_1 - c \cdot q_1 = [P(q_1 + q_2) - c] \cdot q_1$$

$$\pi_2(q_1, q_2) = P(q_1 + q_2) \cdot q_2 - c \cdot q_2 = [P(q_1 + q_2) - c] \cdot q_2$$

We can write this problem in the following way:

$$q_1^* = \arg\max_{q_1 \in [0,a)} [a - (q_1 - q_2^*) - c] \cdot q_1$$

$$q_2^* = \arg\max_{q_1 \in [0,a)} [a - (q_1^* - q_2) - c] \cdot q_2$$

Notice that, given what firm 1 produces, firm 2 produces the quantity that maximizes its profits (and vice-versa). Therefore to firm 1, q_2^* is a parameter (and vice-versa).

Taking the FOCs:

$$\pi_1'(q_1, q_2^*)|_{q_1 = q_1^*} = a - 2q_1^* - q_2^* - c = 0$$

$$\pi_2'(q_1, q_2^*)|_{q_2 = q_2^*} = a - q_1^* - 2q_2^* - c = 0$$

Then, solving the system:

$$\begin{cases} a - 2q_1^* - q_2^* - c = 0 \\ a - q_1^* - 2q_2^* - c = 0 \end{cases}$$

$$\Rightarrow q_1^* = q_2^* = \frac{a - c}{3}$$

This is the quantity produced by each firm in a non-cooperative setup.[2]

[2]Notice that $q_1^* + q_2^* > a$ cannot be a NE. Indeed, this implies that $q_1^* > 0$ or $q_2^* > 0$ or both. Take $q_1^* > 0$. Then $\pi_1(q_1, q_1) = -cq_1^* < 0$, which can be strictly improved upon by not producing. The same holds for $q_2 > 0$.

Example 6: Matching Pennies. A zero-sum game without Pure Strategies NE

A famous class of games is that defined *Zero-Sum Games* (or Constant-Sum Games). These are games of pure conflict, where what a player yields is exactly what the other player loses.

An example is offered by "Matching Pennies" (or Head and Tail). Let's see the following matrix:

Player 1 | Player 2	Head	Tail
Head	-1,1	1,-1
Tail	1,-1	-,1

Looking for NE, if player 1 plays **Head**, player 2 plays **Tail**. If player 1 plays **Tail**, the other plays **Head**. Then, there are no Pure Strategies NE. Let's look for mixed strategies.

If we take $\sigma_2 = (\frac{1}{2}, \frac{1}{2})$, and compute the utility for player 1:

$$u_1(H, \sigma_2) = \frac{1}{2} \cdot -1 + \frac{1}{2} \cdot 1 = 0$$

$$u_2(T, \sigma_2) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot -1 = 0$$

The same for player 2 and $\sigma_1 = (\frac{1}{2}, \frac{1}{2})$. Therefore, there is a (mixed strategies) NE, which is $\sigma^* = (\sigma_1^*, \sigma_2^*)$.

2.4.2 Nash Equilibrium (general)

Now, we define the concept of NE of mixed strategies. Notice that the idea is the same; namely, unilateral deviations are not optimal.

Definition 2.4.3 (Nash Equilibrium)**.** A strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a NE $\forall i$ and $\forall s_i \in S_i$ if:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*).$$

Equivalently: σ^* is a NE if and only if for all $i = 1, \ldots, n$, every $s_i \in Supp(\sigma_i^*)$ belongs to $BR_i(\sigma_{-i}^*)$.

Equivalently: σ^* is a NE if and only if for all $i = 1, \ldots, n$:

$$u_i(\sigma_i^*, \sigma_{-i}^*) = u_i(s_i, \sigma_{-i}^*) \;\; \forall s_i \in S_i$$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \;\; \forall s_i \in S_i - Supp(\sigma_i^*)$$

Then we can state this fundamental proposition.

Theorem 2.4.1 (Existence of Nash Equilibrium)**.** *If $G = (S_1, \dots, S_n, u_1, \dots, u_n)$ is finite (i.e., for any S_i, it has finitely many actions), then a Nash Equilibrium (possibly in Mixed Strategies) always exists.*

Proof. (missing) □

Notice that the finiteness of games is just a sufficient condition, not a necessary one. Indeed games can have a NE even if they are not finite (an example is Cournot Duopoly). Besides, this theorem just says that a NE equilibrium exists, not how many or which type.

Example 7: Battle of Sexes III

Let's go back to the Battle of Sexes.

Alice | Bob	Opera	Football
Opera	3,1	0,0
Football	0,0	1,3

Above, we have found and graphed $BR_A(\sigma_B)$. Let's now look for $BR_B(\sigma_A)$. Then:

$$u_B(\sigma_A, Op) = \alpha$$
$$u_B(\sigma_A, F) = 3 - 3\alpha$$

Then:

$$\alpha \geq 3 - 3\alpha$$
$$-3 \geq 4\alpha$$
$$\alpha \geq \frac{3}{4}$$

Therefore:

$$BR_B(\sigma_A) = \begin{cases} \beta = 1 \text{ if } \alpha > \frac{3}{4} \\ \beta \in [0,1] \text{ if } \alpha = \frac{3}{4} \\ \beta = 0 \text{ if } \alpha < \frac{3}{4} \end{cases}$$

Graphically see Figure 3. The three points where $BR_A(\sigma_B)$ and $BR_B(\sigma_A)$ crosses are all the NE (mixed and pure strategies) of this game (2 pure strategies NE, and one mixed strategies NE).

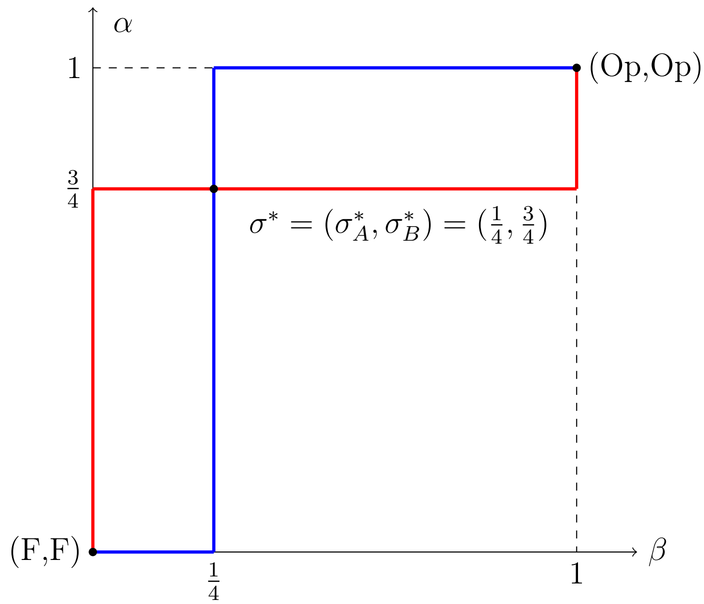


Figure 2.2: NE in the Battle of Sexes

2.4.3 Nash Equilibrium and the removal of dominated strategies

NE strategies survive the iterative removal of all strictly dominated strategies. Recall that the outcome of IRSDS can be written as $S_i^\infty = \bigcap_{k=0}^{\infty} S_i^k$. Therefore, all the NE $\subseteq S_i^\infty$.

Furthermore, we can state the following result.

Theorem 2.4.2. *Suppose $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a NE. Then,$\forall i, \quad \forall s_i \in S_i$ such that $\in Supp(\sigma^*)$:*

$$Supp(\sigma^*) \subseteq S_i^\infty$$

Proof. The proof is by contradiction. Assume $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ is a NE, but $Supp(\sigma_j^* \subseteq S_j^\infty$ for some j. Then suppose also $s_i \in S_i^\infty$ is the first action in $Supp(\sigma_1^*) \cup Supp(\sigma_2^*) \cup \dots \cup Supp(\sigma_n^*)$ to be deleted in the process of IRSDS.

If $S - i$ is deleted in round k, then $Supp(\sigma_j^*) \subseteq S_j^{k-1}$ for all $j = 1, \dots, n$. S_i deleted in round k means that it exists a $\sigma_i \in \Delta(S_i^{k-1}$ such that $u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in \bigtimes S_{j \neq i}^{k-1}$.

This implies that:

$$u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in \bigtimes_{j \neq i} Supp(\sigma_j^*)$$

Because $Supp(\sigma_j^*) \subseteq S_j^{k-1}$. Then:

$$\underbrace{\sum_{s_{-i}\in S_j}\Big(\prod_{j\neq i}\sigma_j^*(s_j)\Big)u_i(\sigma_i', s_{-i})}_{u_i(\sigma_i,\sigma_{-i}^*)} > \underbrace{\sum_{s_{-i}\in S_j}\Big(\prod_{j\neq i}\sigma_j^*(s_j)\Big)u_i(s_i, s_{-i})}_{u_i(\sigma_i^*,\sigma_{-i}^*)}$$

S_i is not a BR, so we have reached a contradiction. □

Notice that this result is extremely useful since it allows to reduce the set of all possible NE in the game, by just looking at those strategies that survive IRSDS. The result above has a fundamental corollary.

Corollary 2.4.2.1. *Suppose $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ is finite, and $s^* = (s_1^*, \ldots, s_n^*)$ is the unique strategy profile that survives IRSDS. Then s_n^* is the unique NE.*

Still, to apply this result, we have two limitations. First, we need strict domination; second, we need finite games.

For an example of what happens in the case of Weak Domination, see the following game:

Player 1 | Player 2	Left	Right
Up	1,1	0,0
Down	0,0	0,0

In this game we have two NE, (**Down**,**Right**) and (**Up**,**Left**). Notice also that both **Down** and **Right** are weakly dominated.

In the case of pure strategies (and also in this case), we can define a Strict NE.

Definition 2.4.4. A Nash Equilibrium $s^* = (s_1^*, \ldots, s_n^*)$ is strict if $\forall i$, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)\ \ \forall s_i \in S_i^*$

Besides, the following proposition links NE and the Iterated elimination of weakly dominated strategies.

Proposition 8. *Suppose $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ is finite, and $S^* = (S_1^*, \ldots, S_n^*)$ is the unique strategy profile that survives iterated deletion of weakly dominated strategies. Then S_n^* is a NE.*

Proof. The proof is by contradiction. Suppose S^* is not a NE. Then, it exists i and s_i such that $u_i(s_i^*, s_{-i}^*) < u_i(s_i, s_{-i}^*)(*)$. Notice that to be a not NE, it is sufficient that a pure strategy is better).

But S_i is deleted in some round k, so it exists $\sigma_i \in \Delta(S_i^{k-1}$ such that:

$$u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})\ \ \forall s_{-i} \in \bigtimes_{j\neq i} S_j^{k-1}$$

This implies:

$$u_i(\sigma_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$$

s^*_{-i} survives to all eliminations. Then it exists $s'_i \in S_i^{k-1}$ such that $u_i(s'_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})$ (**).

Then we have:

$$u_i(s^*_i, s^*_{-i}) < u_i(s'_i, s^*_{-i})$$

If $s^*_i = s'_i$, then we have reached a contradiction. If $s'_i \neq s^*_i$, we continue. s'_i is deleted in some round $k' > k$. This implies that it exists a $\sigma'_i \in \Delta(S_i^{k-1}$ such that $u_i(\sigma'_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \;\; \forall s_{-i} \in \bigtimes_{j \neq i} S_j^{k-1}$. Then:

$$u_i(\sigma'_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i})(***)$$

Then it exists $s_i" \in S_i^{k'-1}$ such that: $u_i(s_i", s^*_{-i}) \geq u_i(s'_i, s^*_{-i})$(****). This implies $u_i(s^*_i, s^*_{-i}) < u_i(s_i", s^*_{-i})$.

If $s^*_i = s_i"$, we have reached a contradiction. Otherwise, let's continue with another round. Still, since the game is finite, a contradiction will be reached at a certain point. This completes the proof. □

Example 8: All Pay Auction

(Missing: see notes)

Example 9: Hotelling's Model

This is a model that shows the idea of *Spatial Competition*. Players compete in choosing a location in a segment between extremes $[0, 1]$ (we assume that there is a continuum of consumers). There are two sellers, many consumers, and each buys only one unit of goods from the sellers (the goods are the same, and their price is normalized to 1). Each consumer buys from the closest seller. Let's find all the pure NE of the game.

The setups of the model are the following: the strategy space S_j is $[0, 1]$. The payoff for each vendor coincides with their profits, and, therefore, with the location they chose:

$$u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2} \text{ if } s_i < s_j \\ \frac{1}{2} \text{ if } s_i = s_j \\ 1 - \frac{s_i+s_j}{2} \text{ if } s_i > s_j \end{cases}$$

We can see that choosing a location different from that of the other vendor is not a NE. Indeed each can find a profitable deviation.

$$u_i(s_i, s_j) = \frac{s_i + s_j}{2} < \frac{s_i + \epsilon + s_j}{2} = u_i(s'_i, s_j)$$

Thus, in a NE, both players must choose the same location. Still, this is not sufficient. Indeed assume they pick the same location in a random point between $[0, 1]$ different from the median point. Then, each has the incentive to deviate. By moving slightly toward the center, he can obtain more than $\frac{1}{2}$. Then, the unique NE is the strategy where both the sellers chose the same location, and this coincides with the median point.

Indeed, in this case:

$$u_i\left(s_1 = \frac{1}{2}, s_j = \frac{1}{2}\right) = \frac{1}{2} > \frac{s_i' + \frac{1}{2}}{2} = u_i\left(s_i', s_j = \frac{1}{2}\right) \text{ where } s_i' < \frac{1}{2}$$

$$u_i\left(s_1 = \frac{1}{2}, s_j = \frac{1}{2}\right) = \frac{1}{2} > 1 - \frac{s_i' + \frac{1}{2}}{2} = u_i\left(s_i', s_j = \frac{1}{2}\right) \text{ where } s_i' > \frac{1}{2}$$

Still, notice that this is not true anymore in the case of three vendors. Here we have three cases.

First, each can choose a different location, say $s_1 < s_2 < s_3$. Each player has the incentive to deviate. For instance, s_1 can choose a strategy closer to s_2.

Second, two players pick the same location, and the third one does not. Say $s_1 = s_2 = s < s_3$. In this case, s_3 can pick any location closer to $s_1 = s_2$ (that is, comprised in the interval (s_3, s) if $s_3 > s$ and in the interval (s, s_3) if $s_3 < s$.

The third case is that of all players picking the same location s. In this case, each gets $\frac{1}{3}$. Assume s is not the median point. Then, player 1 can choose the median point, assuring a payoff greater than $\frac{1}{2} > \frac{1}{3}$. Suppose, instead, that s is the median point. Still, each player, say 1, has the incentive to deviate.

Example 10: The problem of the Commons

The main idea is that, if players respond to private incentives only, then public goods will be underprovided and public resources overutilized.

Consider a village with n farmers. Each farmer has some goats, where the number of goats of farmer i is denoted by g_i and therefore, the total number of goats is $G = g_1 + \cdots + g_n$. Each farmer gets a revenue depending on the total number of goats in the field (i.e., more goats in the field, less grass for each goat). This can be written as:

$$v(G) = \begin{cases} v(G) > 0 \text{ if } G < G_{max} \\ 0 \text{ if } G > G_{max} \end{cases}$$

The basic idea is simple. There is a max number of goats that can eat the grass. Once reached that number, the value of taking a goat to the field is 0. Furthermore, the revenue function is concave and decreasing (i.e., $v'(G) < 0$ and $v"(G) < 0$). Finally, each goat generates a positive cost for the farmer.

The payoff function can be written as:

$$u_i(g_i, g_{-i}) = v(G) \cdot g_i - c \cdot g_i$$

The set of strategies is $S_i = [0, G_{max}]$ for all $i = 1, \ldots, n$. Indeed, playing $g_i > G_{max}$ is always dominated by playing 0 (in simple words, bringing a goat in the field when the max number has been reached is always dominated by not bringing a goat).

For each farmer the problem is that of maximizing her payoff. Then:

$$\max_{q_i} v(G) \cdot g_i - c \cdot g_i$$

The FOC of the i-th farmer is:

$$v(g_1^* + \cdots + g_i^* + \cdots + g_n^*) + g_i \cdot v'(g_1^* + \ldots g_i^* + \cdots + g_n^*) - c = 0$$

This can be written as:

$$v(G^*) + g_i^* v'(G^*) - c = 0$$

Multiply by n:

$$n \cdot v(G^*) + \underbrace{n \cdot g_i^*}_{G^*} v'(G^*) - n \cdot c = 0$$

Divide by n:

$$v(G^*) + \frac{G^*}{n} v'(G^*) - c = 0$$

This is the optimal output for each player.

Let's now suppose that there is a *Benevolent Planner*. He is called to choose the G that solves the following problem:

$$\max_{0<G<\infty} v(G) \cdot G - c \cdot G$$

Taking the FOC (and denoting as $\tilde{G}$ the G that solves the FOC):

$$\tilde{G} \cdot v'(\tilde{G}) + v(\tilde{G}) - c = 0$$

The interesting problem is now that of determining which one is bigger: $\tilde{G}$ or G^*?

We want to show that $\tilde{G} < G^*$. Let's rewrite the two optimal, that of individuals and the collective one) as:

$$v(G^*) + \frac{G^*}{n} v'(G^*) = c$$

$$\tilde{G} \cdot v'(\tilde{G}) + v(\tilde{G}) = c$$

Then:

$$v(G^*) + \frac{G^*}{n} v'(G^*) = \tilde{G} \cdot v'(\tilde{G}) + v(\tilde{G}) = c$$

Assume (by contradiction) that: $\tilde{G} \geq G^*$. This implies that $\tilde{G} > \frac{G^*}{n}(*)$. Then, since the revenue function is decreasing:

$$v(G^*) \geq v(\tilde{G})(**)$$

Furthermore, since the revenue function is strictly concave:

$$0 > v'(G^*) > v'(\tilde{G})(***)$$

From (*) and (***), we have:

$$v'(G^*)\frac{G^*}{n} > v'(\tilde{G}) \cdot \tilde{G}$$

From (**)

$$\underbrace{v(G^*) + v'(G^*)\frac{G^*}{n}}_{= c \text{ from the FOC}} > \underbrace{v(\tilde{G} + v'(\tilde{G}) \cdot \tilde{G}}_{= c \text{ from the FOC}}$$

Since we have $c > c$, we have reached a contradiction. Therefore $G^* > \tilde{G}$.

This means that the NE equilibrium of this non-cooperative game is greater than the social optimum.

How to interpret this result? The economic intuition is that the common resource is over-utilized. Indeed, from the FOC, for each player we can write:

$$v(G^*) + \frac{G^*}{n}v'(G^*) - c = 0$$

Then the Marginal Revenues is $v(G^*)$, and the Marginal Costs is $c - v'(G^*) \cdot \frac{G^*}{n}$. This is greater than zero (since $v'(G^*)$ is greater than zero). So the overall quantity is positive. Notice that The marginal cost is less than the marginal cost for society, since, in the latter case, this is $c - v'(\tilde{G}) \cdot \tilde{G}$. Therefore, bringing an extra goat produces greater damage to society than to a single farmer. This is a problem of *Negative Externality.*

Chapter 3

Games in Strategic Form with Incomplete Information: Bayesian Games

In a game with complete information, the players' payoffs are *Common Knowledge*. In games of incomplete information, instead, at least one player is uncertain about the other's payoff functions. These games have been studied extensively by John Harsanyi in the 1960s. He came up with the following idea: players have been endowed with different *types*, chosen randomly by nature. An example is the following: assume that one can like chocolate or strawberries. Then, in this framework, there exist two different types, say, that who likes chocolate and that who does like strawberries.

3.1 Fundamentals

Before presenting the fundamentals of these games, notice that they are common knowledge. Uncertainty is only about the payoffs.

The setup of these games is the following. We have n-players $i = 1, \ldots, n$. Associated to each player is the *Set of types of the player*, T_i. Each type t_i belongs to this set. The *Set of type profiles* is:

$$T = \bigtimes_{i=1}^{n} T_i$$

Each $t = (t_1, \ldots, t_n)$ belongs to T. We can also write $T_{-i} = \bigtimes_{i \neq j}^{n} T_j$. Then $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ belongs to T_{-i}. We can write, for simplicity, $t = (t_i, t_{-i})$.

We also define the *Set of Actions* for player i, as A_i. And the set of all actions as: $\bigtimes_{i=1}^{n} A_i$ and $A_{-i} = \bigtimes_{j \neq i} A_j$ is the set of actions of all opponents to player i.

The *Payoff Function* is:

$$u_i : A \times T \longrightarrow \mathbb{R}$$

Finally, we can also define a probability distribution over the different types, $p \in \Delta(T)$ (this is called *Common Prior*).

Therefore, we can define a Games in Strategic Form with Incomplete Informaton (or *Bayesian Games* as follows:

$$\Gamma = (T_1, \dots, T_n, p, A_1, \dots, A_n, u_1, \dots, u_n)$$

We can now outline the timing of a static Bayesian Game as originally proposed by Harsanyi.

- Nature chooses a type profile for each player, $t = (t_1, \dots, t_n)$ according to probability p.
- Each player i learns upon t_i
- The players choose their actions simultaneously
- payoffs are received

Since each player knows only her type, the types of others are private information. Notice that the case of games with complete information can be interpreted as a special case of Bayesian Games, namely where T_i is a singleton.

Let's see an example with $n = 2$. Each player has the following set of actions:

$$A_1 = \{U, D\} \text{ and } A_2 = \{L, R\}$$

And the following set of types:

$$T_1 = \{\tilde{t}_1, \hat{t}_1\} \text{ and } T_2 = \{\tilde{t}_2, \hat{t}_2\}$$

The probability distributions for each possible case can be:

$$p(\tilde{t}_1, \tilde{t}_2) = \frac{3}{10}$$
$$p(\hat{t}_1, \hat{t}_2) = \frac{4}{10}$$
$$p(\tilde{t}_1, \hat{t}_2) = \frac{2}{10}$$
$$p(\hat{t}_1, \hat{t}_2) = \frac{1}{10}$$

Each of the four combinations above can be associated with a game matrix. Still, note that these are *all part of the same game.*

Each player knows her type and must guess the type of the opponent. Namely, if player 1 is of type $\tilde{t}_1$, what is the probability that 2 be of type $\tilde{t}_2$? This can be written as:

$$p(\tilde{t}_2|\tilde{t}_1) = \frac{p(\tilde{t}_1,\tilde{t}_2)}{\underbrace{p(\tilde{t}_1,\tilde{t}_2)+p(\tilde{t}_1,\hat{t}_2)}_{p(\tilde{t}),\text{ the Marginal probability of } t_i}} = \frac{\frac{3}{10}}{\frac{3}{10}+\frac{2}{10}} = \frac{3}{5}$$

Or, more in general:

$$p(t_{-i}|t_i) = \frac{p(t_{-i},t_i)}{p(t_i)}$$

Where the Marginal Probability of i is: $p(t_i) = \sum_{t_{-i}\in T_{-i}} p(t_{-i},t_i)$. Assuming that the probabilities are independent, $p(t_1,\dots,t_n)$ can be written as $\prod_{i=1}^{n} p(t_i)$.

Therefore, we can also write:

$$p(t_{-i}|t_i) = \frac{\prod_{i=1}^{n} p(t_i)}{\sum_{t_{-i}\in T_i} p(t_{-i},t_i)}$$

3.2 Pure Strategies and Mixed Strategies

3.2.1 Pure Strategies

Different types do different things. Pure strategies specify an action for any type. This can also be seen as a contingent plan of action for any type, given that each player knows her type. It is a mapping from types to actions and can be written as:

$$s_i : T_i \longrightarrow A_i$$

Let's see an example. $T_1 = \{\tilde{t}_1, \hat{t}_1\}$ is the set of types for player 1 and $A_1 = \{U, D\}$ is the set of actions. We can define the following pure strategies:

$$s_1(\tilde{t}_1) = U$$
$$s_1(\hat{t}_1) = D$$
$$s_1(\tilde{t}_1) = s_1(\hat{t}_1) = U$$
$$s_1(\tilde{t}_1) = D$$
$$s_1(\hat{t}_1) = U$$
$$s_1(\tilde{t}_1) = s_1(\hat{t}_1) = D$$

S_i is *Set of all pure strategies.* The cardinality of S_i is given by $|A_i|^{|T_i|}$, the number of pure strategies raised to the number of possible types.

3.2.2 Mixed Strategies

Mixed strategies can be defined in two different but equivalent, ways.

1. If S_i is the set of pure strategies, we can randomize over this set. Then, $\sigma_i \in Delta(S_i)$. In the example above we have four pure strategies:

   $$S_i = \{(U,U),(U,D),(D,U),(D,D)\}$$

   Then we can have the following mixed strategies: $\sigma_1(U,U) = 0.4$, $\sigma_1(U,D) = 0.3$, $\sigma_1(D,U) = 0.2$, $\sigma_1(D,D) = 0.1$.

2. For each type we randomize. Namely we construct a function:

   $$\beta : T_i \longrightarrow \Delta(A_i)$$

   β is called *Behavioral Function.* Then we can have:

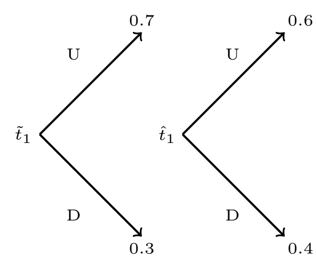


Figure 3.1: Behavioral Strategies

Notice that (1) and (2) are different mathematical objects. But still, they are equivalent.

3.3 Bayesian Nash Equilibrium

Suppose that there is a set of actions $a = (a_1, \ldots, a_n)$ and of types $t = (t_1, \ldots, t_n)$. Player j randomizes and choose a_j with probability $\alpha_j(a_j)$. Therefore, we can write the expected payoff from this randomization as:

$$u_i(\alpha_1, \ldots, \alpha_n, t_1, \ldots, t_n) = \sum_{\alpha_1,\ldots,\alpha_n} \left(\prod_{j=1}^{n} \alpha_j(a_J) \right) u_i(a,t)$$

Now we can define dominance in Bayesian Games.

Definition 3.3.1. A mixed strategy σ is stictlu dominated if it exists a $t_i \in T_i$ and $\alpha_i \in \Delta(A_i)$ such that:

$$\sum_{t_{-i}} p(t_{-i}|t_i)u\Big(s_1(t_1), \dots, s_{i-1}(t_{i-1}), \alpha_j, s_{i+1}(t_{i+1}, \dots, s_n(t_n), t_{-i}, t_i)\Big) >$$
$$\sum_{t_{-i}} p(t_{-i}|t_i)u\Big(s_1(t_1), \dots, s_{i-1}(t_{i-1}), \beta_i(t_1), s_{i+1}(t_{i+1}), \dots, s_n(t_n), t_{-i}, t_i)\Big)$$

For all pure strategy profile $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$, when $s_j : T_j \longrightarrow A_j$.

This definition can be extended to the case of weak domination when $\geq$ and at least one strict inequality.

Now, we can also define the Bayesian Nash Equilibrium (BNE):

Definition 3.3.2. Behavioral strategies $(\beta_1^*, \dots, \beta_n^*)$, where $\beta_j : T_j \longrightarrow \Delta(A_j)$ is a Bayesian Nash Equilibrium if, $\forall i = 1 \dots, n$,$\forall t_i \in T_i$ and $\forall a_i \in A_i$:

$$\sum_{t_{-i}} p(t_{-i}|t_i)u_i\Big(\beta_1(t_1), \dots, \beta_{i-1}(t_{i-1}), \beta_i(t_i), \beta_{i+1}(t_{i+1}), \dots, \beta_n(t_n), t_{-i}, t_i\Big) \geq$$
$$\sum_{t_{-i}} p(t_{-i}|t_i)u_i\Big(\beta_1(t_1), \dots, \beta_{i-1}(t_{i-1}), \alpha_i, \beta_{i+1}(t_{i+1}), \dots, \beta_n(t_n), t_{-i}, t_i\Big)$$

Notice that $p(t_{-i}|t_i) = \frac{p(t_{-i},t_i)}{p(t_i)}$. Therefore, by multiplying both sides by $p(t_i)$, we can write the definition above in terms of *Joint Probabilities* and not conditional probabilities without change.

Example 11: Cournot with Private Information

Two firms face the problem of setting the quantity to produce in order to maximize their profits. We can write the inverse demand function as:

$$P(q_1 + q_2) = \begin{cases} a - (q_1 + q_2) & \text{if } q_1 + q_2 \leq a \\ 0 & \text{otherwise} \end{cases}$$

But now, firm 2 has private information about its costs. Then the cost functions for each firm are:

$$c_1(q_1) = c \cdot q_1 \quad c > 0$$

and

$$c_2(q_2) = c_H \cdot q_2 \quad \text{with probability } \theta$$
$$c_2(q_2) = c_L \cdot q_2 \quad \text{with probability } 1 - \theta$$

with $0 < c_H < c_L$ and $\theta \in [0, 1]$. The set of types, for each firm, are the following: $T_1 = \{t_1\}$ (since firm 1 has one type only) and $T_2 = \{t_L, t_H\}$. Then we have:

$$p(t_1, t_H) = \theta \text{ and } p(t_1, t_L) = 1 - \theta$$

The payoff functions for each firm are the following. Starting with Firm 1:

$$\pi_1(q_1, q_2, t_1, t_L) = \pi(q_1, q_2, t_1, t_H) = (P(q_1 + q_2) - c) \cdot q_1$$

Notice that the two functions for t_H and t_L are equal because the type of the opponent does not directly affect the payoff of Firm 1.

Let's see now for Firm 2. In this case, we have two different payoff functions, depending if the firm is of type t_H or t_L.

$$\pi_2(q_1, q_2, t_1, t_L) = (P(q_1 + q_2) - c_L) \cdot q_2$$

$$\pi_2(q_1, q_2, t_1, t_H) = (P(q_1 + q_2) - c_H) \cdot q_2$$

Looking for Pure Strategies BNE, we need to specify the strategies for players 1 and 2. For player 1, it is one number q_1^*. For player 2 are two numbers (q_L^*, q_H^*).

$$q_1^* = \arg\max_{q_1 \in [0,a)} \theta[a - q_1 - q_H^* - c] \cdot q_1 + (1 - \theta)[a - q_1 - q_L^* - c] \cdot q_1$$

$$q_L^* = \arg\max_{q_2 \in [0,a)} (a - q_1^* - q_2 - c_L) \cdot q_2$$

$$q_H^* = \arg\max_{q_2 \in [0,a)} (a - q_1^* - q_2 - c_L) \cdot q_2$$

Taking the FOCs, and solving the system of three equations and three unknowns, we have:

$$\begin{cases} \theta(a - 2q_1^* - q_H^* - c) + 1 - \theta(a - 2q_1^* - q_L^* - c) = 0 \\ a - q_1^* - 2q_H^* - c_H = 0 \\ a - q_1^* - 2q_L^* - c_L = 0 \end{cases}$$

And

$$\begin{cases} q_1^* = \frac{a - 2c - \theta c_H + (1-\theta)c_L}{3} \geq 0 \\ q_L^* = \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L) \geq 0 \\ q_H^* = \frac{a - 2c_H + c}{3} - \frac{(1-\theta)}{6}(c_H - c_L) \geq 0 \end{cases}$$

Notice that $q_L^* = q_H^* + \frac{1}{2}(c_H - c_L) > q_H^*$.

Since all q_i are greater than zero, and all prices are greater than zero, the system above is the Bayesian Nash Equilibrium of the game.

Example 12: Battle of Sexes IV

Let's return to the Battle of Sexes, seen previously. Recall that the mixed strategy NE was: $\sigma_A^* = (\frac{3}{4}, \frac{1}{4})$ and $\sigma_B^* = (\frac{1}{4}, \frac{3}{4})$. Suppose now that, for some shock, both Alice's and Bob's payoffs change, such that the new Payoff Matrix is the following:

Alice | Bob	Opera	Football
Opera	$3 + t_A$,1	0,0
Football	0,0	1, $3 + t_B$

This change is *Private Information.* Furthermore, we assume that t_A and t_B are independent draws from a uniform distribution $[0, x]$. For all $y \in [0, x]$ then $P(t_A \leq y) = P(t_B \leq y) = \frac{y}{x}$.

The set-up of the game are $n = 2$ (the number of players). The sets of actions are $A_A = A_B = \{Op, F\}$. The sets of types are $T_A = T_B = [0, x]$.

Let's find the pure strategies BNE. First, notice that each player's strategy can be written as follows:

$$s_A : [0, x] \longrightarrow \{Op, F\}$$

$$s_B : [0, x] \longrightarrow \{Op, F\}$$

Fix s_B, the idea is that Alice goes to Opera if t_A exceeds a certain value, say $\tilde{t}_A$. And the same does Bob with $\tilde{t}_B$. So that we can write:

$$S_A^*(t_A) = \begin{cases} Op & \text{if } t_A \leq \tilde{t}_A \\ F & \text{if } t_A > \tilde{t}_A \end{cases}$$

and

$$S_B^*(t_B) = \begin{cases} Op & \text{if } t_B \leq \tilde{t}_B \\ F & \text{if } t_B > \tilde{t}_B \end{cases}$$

Let's find $\tilde{t}_A$ and $\tilde{t}_B$. When $t_A = \tilde{t}_A$, Alice is indifferent between Opera and Football. Then we can write:

$$\frac{\tilde{t}_B}{x}(3 + \tilde{t}_A) = 1 - \frac{\tilde{t}_B}{x}$$

This is Alice's expected payoff when her type is $\tilde{t}_A$, and Bob plays Opera. For Bob, when $t_B = \tilde{t}_B$ we write:

$$\frac{\tilde{t}_A}{x}(3 + \tilde{t}_B) = 1 - \frac{\tilde{t}_A}{x}.$$

$\tilde{t}_A$ and $\tilde{t}_B$ must satisfy the following system (multiplying each equation above by x).

$$\begin{cases} 3\tilde{t}_B + \tilde{t}_B \cdot \tilde{t}_A = x - \tilde{t}_B \\ 3\tilde{t}_A + \tilde{t}_B \cdot \tilde{t}_A = x - \tilde{t}_A \end{cases}$$

$$\begin{cases} 4\tilde{t}_B = x - \tilde{t}_B \cdot \tilde{t}_A \\ 4\tilde{t}_A = x - \tilde{t}_B \cdot \tilde{t}_A \end{cases}$$

Since $\tilde{t}_A = \tilde{t}_B$, we can write:

$$4\tilde{t}_A = x - (\tilde{t}_A)^2$$
$$= \tilde{t}_A = \frac{-4 + \sqrt{16 + 4x}}{2}$$

Where $\tilde{t}_A = \tilde{t}_B \in [0, x]$.

The probability that each player plays his favorite strategy, p(A plays Op) and p(B plays F), is $1 - \frac{\tilde{t}_A}{x}$, namely:

$$1 - \frac{-4 + \sqrt{16 + 4x}}{2x}$$

To conclude, let's notice an important result established originally by Harsanyi. As $x \to 0$, we obtain $\frac{0}{0}$. Therefore, by applying the Rule of de l'Hopital[1] we obtain $\frac{1}{4}$. Therefore, as $x \to 0$, we have:

$$1 - \frac{1}{4} = \frac{3}{4}$$

This is exactly the mixed strategy NE in the original game of complete information (*Harsanyi's Purification*).

Example 13: First Price Auction with Independent Private Values

In this auction, players submit bids, and the player with the highest bid wins and pays his bid. *Independent* refers to the distribution of types. *Private* refers to a property of the payoffs, namely v_i depends on t_i and not on t_{-i}.

The setup of the model is the following. There are $n-$players. $A_i = \mathbb{R}_+$ and $T_i = [0, v]$. $v_i \in [0, v]$ is i's type. Types are identically independently distributed. The distribution of types is indicated by a cumulative distribution function $F(.)$, where $f(0) = 0$ and $F(v) = 1$. Then, $\forall \tilde{v} \in [0, v]$ we can write $p(v_i \leq \tilde{v}) = F(\tilde{v})$.

We can write each player's payoff function as:

$$u_i(b_1, \ldots, b_n, v_1, \ldots, v_n) = \begin{cases} v_i - b_i \text{ if } b_i > b_j \text{ for } i \neq j \\ \frac{v_i - b_i}{k+1} \text{ if } i \text{ ties with } k \text{ opponents} \\ 0 \text{ if } b_j > b_i \text{ for } j \neq i \end{cases}$$

[1]This rule assume that if $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, and $g'(x) \neq 0$, then we can write:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Let's construct a symmetric pure strategy Bayesian Nash Equilibrium. *Symmetric* refers to the fact that all players have the same strategies and the same payoffs.

We can denote each player's strategy as:

$$g_i : [0, v] \longrightarrow \mathbb{R}_+$$

Intuitively we can assume that this function is strictly increasing (that is, if $v_1 > v_2$, then $g(v_1) > g(v_2)$) and continuous. Still, notice that we need to show that such a function exists and has these features.

First notice that $g(v_i) \leq v_i$. Indeed, the function $g(.)$ must remain below the 45 line. To see this, assume it is not. Then, we have $g(\tilde{v}) > \tilde{v}$. If we write $(F(\tilde{v}))^{n-1} \cdot (\tilde{v})$, this is less than zero (indeed, the first term, $p(v_i \leq \tilde{v})$ is positive, the second term is negative). But then this is dominated by bidding zero, then $g(.)$ cannot be a NE.

We still do not know if $g(.)$ is continuous and strictly increasing. To see this, let's continue to construct $g(.)$.

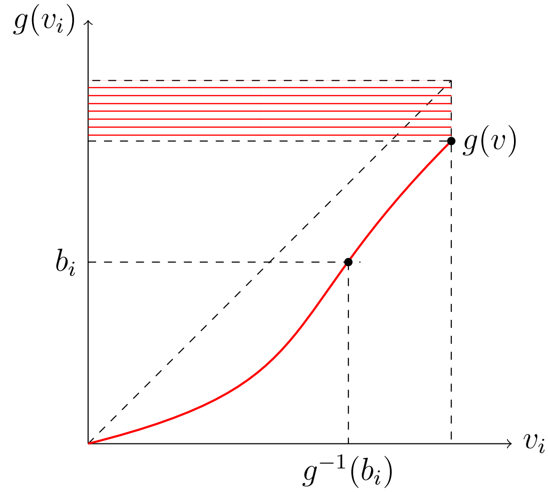


Figure 3.2: A graphical representation of $g(.)$

Assume all the opponents play $g(.)$. Then $\forall v_i \in [0, v]$ the optimal bid is $g(v_i)$.[2] Suppose you choose b_i. Then the expected payoff is:

$$F(g^{-1}(b_i))^{n-1} \cdot (v_i - b_i)$$

And since the optimal bid is $g(v_i)$ we can write:

$$\arg\max_{b_i \in [0,v]} F(g^{-1}(b_i))^{n-1} \cdot (v_i - b_i) = g(v_i) \quad (*)$$

Then we must find the function that solves $(*)$ for every v_i.

[2]Notice from figure 5 that any bid superior to $g(v)$ cannot be optimal, since it is always possible to bid less and still win.

First assume, for the sake of simplicity, that $b_i = g(w_i)$ and $w_i = g^{-1}(b_i)$. Therefore, we can write:

$$v_i = \arg\max_{w_i \in [0,v]} F(w_i)^{n-1} \cdot (v_i - q(w_i))$$

Taking the FOC with respect to w_i, we obtain:

$$(n-1)F(v_i)^{n-2}F'(v)(v_i - g(v_i)) - g'(v_i)F(v_i)^{n-1} = 0 \quad (**)$$

This is a differential equation. Assuming, for simplicity, that $F(v_i) = \frac{v_i}{v}$ and $g(v_i) = \alpha v_i$ with $\alpha > 0$. Then we can rewrite the equation above as:

$$(n-1)\Big(\frac{v_i}{v}\Big)^{n-2}\frac{1}{v}\Big(v_i - \alpha v_i\Big) - \alpha\Big(\frac{v_i}{v}\Big)^{n-1} = 0$$

The solution is:

$$\alpha = \frac{n-1}{n}$$

Therefore, the Bayesian Nash Equilibrium is $\frac{n-1}{n} \cdot v_i$. Notice that as n increases, then $n-1$ increases. So, with more opponents, each player bids more aggressively.

Appendix: General Solution

In general the solution for $(**)$ is:

$$\begin{aligned}
&\frac{1}{F(v_i)^{n-1}} \int_0^{v_i} x(n-1)F(x)^{n-2}F'(x)dx = \\
&= v_i - \frac{1}{F(v_i)^{n-i}} \int_0^{v_i} F(x)^{n-1}dx \\
&\equiv g(v_i)
\end{aligned}$$

The second equation is obtained through integration by parts.[3] This function is continuous and strictly increasing. Furthermore, checking the SOC, it reaches a maximum.

[3]Recap of *Integration by Parts*: Recall that $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$. Taking the integral of each term, we have:

$$\int (f(x) \cdot g(x))' = \int f'(x) \cdot g(x) + \int f(x) \cdot g(x)'$$

Rearranging, we have:

$$\begin{aligned}
&f(x) \cdot g(x) = \int f'(x) \cdot g(x) + \int f(x) \cdot g'(x) = \\
&\int f'(x) \cdot g(x) = f(x) \cdot g(x) - \int f(x) \cdot g'(x)
\end{aligned}$$

Chapter 4

Games in Extensive Form: Dynamic Games

So far, we have analyzed only situations where players move simultaneously. The only difference was about the amount of available information (common knowledge or private information). Let's see now another category of games, those that involve a sequential situation. Namely, one player moves first, and the second moves later.

Let's start with an example. In this game, there are two players, i.e., two firms that compete for a market. One firm is incumbent, and the second must decide if enter the market or not. If the second firm stays out, its payoff is zero. If enters, the incumbent can choose two options: to accommodate, and therefore both the firms share the market, or to fight. In this case, the entrant receives a negative payoff. Such a situation can be represented in the following tree form.

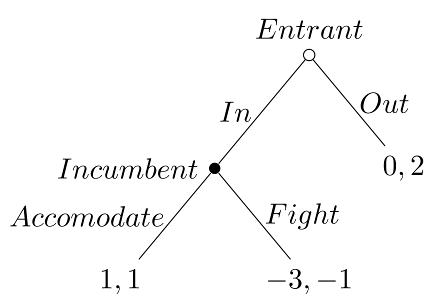


Figure 4.1: An entry game in extensive form

The game above can be represented in the following Normal Form:

Ent | Inc	Fight	Acc
In	-3,-1	1,1
Out	0,2	0,2

Therefore, we see that there are two pure strategies NE: (**In, Acc**) and (**Out, Fight**). However, for player 2 **Fight** is weakly dominated. Besides, looking at the payoffs, Fight is a non-credible threat. This is a characteristic of the game that is not captured in the normal form, but requires that the game is modeled as a sequence of moves.

4.1 Fundamentals

The fundamental feature of Extensive form games is that they can be represented as game trees. We define Y as the set of all the nodes of the tree.X is the set of decision nodes, and Z is the set of terminal nodes. Therefore we have $X \cap Z \neq 0$ and $X \cup Z = Y$. On Y, we introduce a binary relation, q_i, also called *Immediate Procedure Relation.* Then we can write $y_1\ q\ y_2$, i.e., the node y_1 is an immediate predecessor of the node y_2.

Therefore, the tree in Figure 7 can be described as follows:

$$y_1\ q\ y_2, y_2\ q\ y_3, y_3\ q\ y_4, y_4\ q\ y_5.$$

We say that y_0 is a predecessor of y_k if there exist $y_1, \ldots, y_{k-1}$ such that:

$$y_0\ q\ y_1\ q\, y_2 \cdots\ q\ q\ y_k$$

Therefore we say that y_k is a successor of y_0. If $y \in Z$ then we say that y is a terminal node (namely, it has no successors). We say that y is an initial node if it has no predecessors.

The relation q rules out both the possibility that one node is the successor of two different nodes; furthermore also the possibility of cycles.

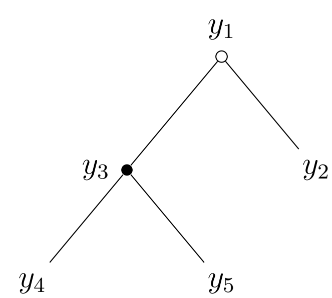


Figure 4.2: A graphical representation of the relation q

Then, we can define a tree as (Y, q). We can outline some properties of a game tree.

- There exists only one initial node
- Every node different from the initial node has a unique immediate predecessor

- For all y, the predecessors of y are completely ordered (i.e., there are no cycles

We define $N = \{0, 1, \ldots, N\}$ as the set of players (where 0 define the "nature". X is the set of decision nodes, then we define ϕ as the set of moving players, namely: $\phi : X \longrightarrow \{0, 1, \ldots, N\}$. The set of actions available to players $\phi(x)$ at nodes x is denoted by $A(x)$.

For each player i we can define a payoff function $u_i : Z \longrightarrow \mathbb{R}$. The set of probability distributions of terminal nodes is $\Delta(Z)$. For every $\pi \in \Delta(Z)$ we can define formally a payoff function as:

$$u_i(\pi) = \sum_{z \in Z} \pi(z) u_i(z)$$

A fundamental notion related to Extensive Form Games is that concerning the information available to different players.

To describe formally this idea, we must introduce the notion of Partition.

Definition 4.1.1 (Partition)**.** Take a set X. A collection $X_1, \ldots, X_n$ of subsets of X is a partition of X if and only if:

- $X_1 \cup X_2 \ldots X_n = X$;
- $X_i \cap X_j = \emptyset$ for all $i \neq j$.

To describe what each player knows in the game, we introduce the partition of the decision nodes, $h_1, \ldots, h_k$. Each $h_j \subseteq X$. Each h_j is called an information set. We also define the set of the information sets, namely $H = \{h_1, h_2 \ldots, h_k\}$, with $h_i \cap h_j = \emptyset$. Every h_j is a collection of nodes. For all $h_i \in H$, and for all $x, x' \in h_i$ we say:

1. x is not a predecessor of x' (and x' is not a predecessor of x)
2. $\phi(x) = \phi(x')$ and $A(x) = A(x')$: each player knows at which node he is.

$\phi(h_i)$ are the players at the information set h_i. $A(h_i)$ are the actions at the information set h_i.

Let's see the following tree as an example:

If player 1 chooses c, then player 2 knows that. Instead, if player 1 chooses a or b, player 2 cannot distinguish between these two choices. Graphically, the information set of each player is represented by a dashed line connecting several decision nodes in the tree.

Notice that we can represent a Normal Form Game in tree form (see Figure 9). Since players move simultaneously in Static Games, we can assume that Player 2 is uncertain about the moves of Player 1. Namely, her information set contains two nodes. Therefore, Games in extensive form are the most general way of representing and solving games.

We can also represent Bayesian Games in Extensive form. First, notice that we can model a decision in the following tree form (Figure 10). Assuming, without loss, that

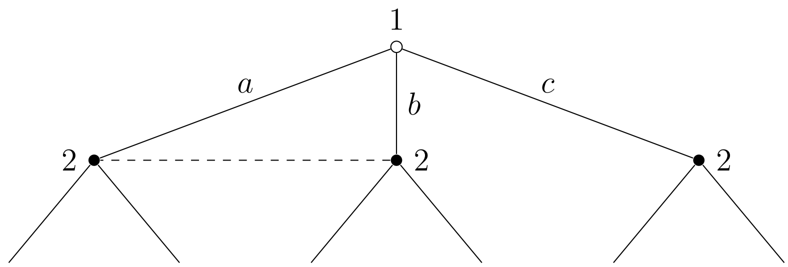


Figure 4.3: Information Sets

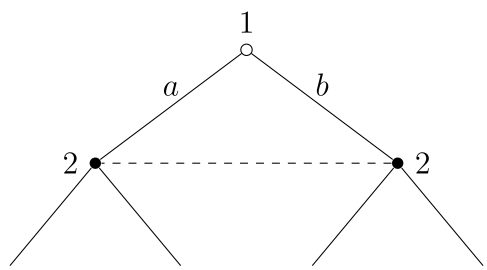


Figure 4.4: A Tree representation of a Normal Form Games

nature moves only at the initial node, we can represent this decision as the game in Figure 11.

More in general, we can represent a Bayesian Game in extensive form, as in Figure 12.

The set of types for each player$i$ is $T_i = \{\tilde{t}_i, \hat{t}_i\}$, the set of actions for player 1 is $A_1 = [A, B]$, for player 2 is $A_2 = [a, b]$. Player 1 knows her type but is uncertain about the type of the opponent (this is represented by the information set of 1). Player 2 knows her type but not the type of 1.

In general, we can define an extensive-form game as:

$$\Gamma = (X, Z, q, N, \phi, A, H, p, u_i, \dots, u_n)$$

Γ is finite if the set Z is *finite*, i.e., if there are finitely many terminal nodes. Γ has *finite horizon* if, for all $z \in Z$, the complete path is finite, namely if we reach any terminal nodes after a finite number of steps). Any finite game has a finite horizon, but the contrary is not always true.

If any information set is a singleton, then this is a game with *Perfect Information* (the simplest example is Chess). On the contrary, games have *Imperfect Information*.

Furthermore, we can also define games with *Incomplete* or *Complete Information*. In the game with *Complete Information*, players start the game with the same private information. All the games with *Incomplete Information* are also games with *Imperfect*

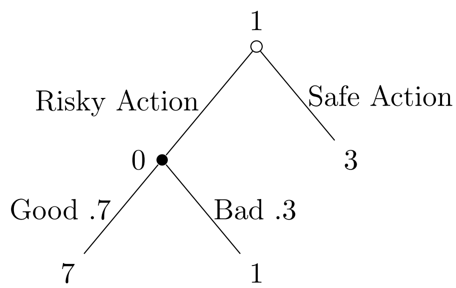


Figure 4.5: A Decision Tree

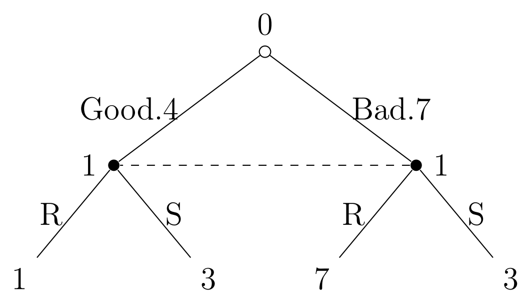


Figure 4.6: A decision as an extensive form game where Nature moves first

Information (the contrary is not true. Games with or *Imperfect Information* are also games with *Complete Information*.

A final category of games is that of games with *Imperfect Recall*. These are games where players forget about their previous moves. A tree for games of this kind is represented in Figure 13. Here player 1's information set comprises nodes from different choices of player 2.

A formal definition of games with *Perfect Recall* is the following.

Definition 4.1.2. If x and x' belong to the same information set, and if $\hat{x}$ is a predecessor of x and $\phi(\hat{x}) = \phi(x)$, then there exists a decision node $\tilde{x}$ (possibly $\hat{x}$ itself) such that $\hat{x}$ and $\tilde{x}$ belong to the same information set, $\hat{x}$ is a predecessor of x', and the action taken at $\hat{x}$ along the path to x coincides with the action taken at $\tilde{x}$ along the path to x'.

Let's see now the idea of Strategies in games of extensive form.

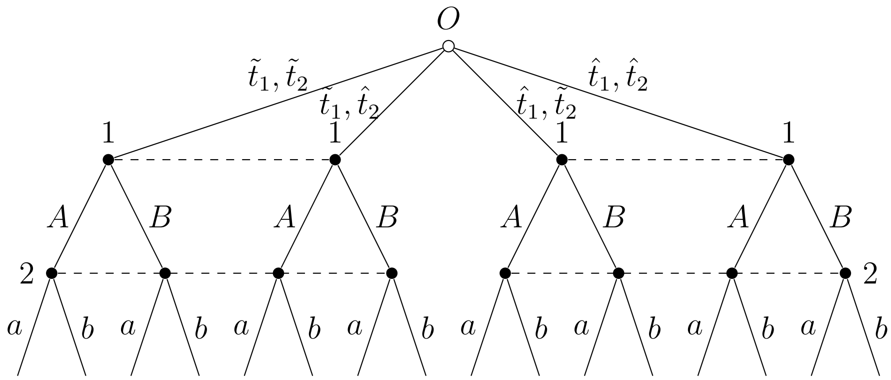


Figure 4.7: A Bayesian Game in Extensive Form

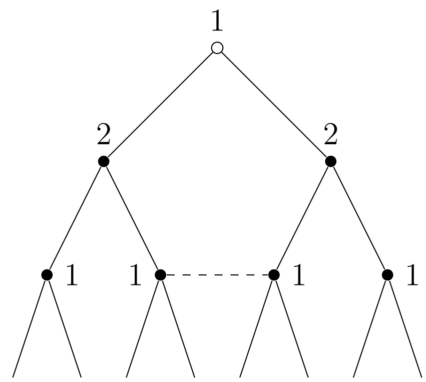


Figure 4.8: A game with imperfect recall

4.2 Pure and Mixed Strategies

4.2.1 Pure Strategies

Let's start with pure strategies. A pure strategy of player i is an action $a \in A(h)$ to every information set h with $\phi(h) = i$. Formally we can write:

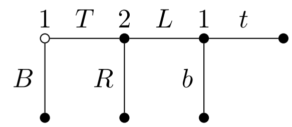
$$s_i : h \in H_i \longrightarrow a \in A(h)$$

Where H_i is the set of information sets of player i. This means that if x, x' belong to the same information set, we cannot say that at node x, player 1 plays a certain strategy, say A, and at x', he plays another strategy, say B.

We can define the set of pure strategies as:

$$\prod_{h \in H_i} |A(h)|$$

Suppose a game as the following:



Then, player 1's pure strategies are ((Bt),(Tb),(Tt),(Bb)). Specifying (Bt) and (Bb) is necessary to determine if player 2's response is optimal.

4.2.2 Mixed Strategies

We can think of mixed strategies in two types:

1. List all the pure strategies and then randomize. Specify a probability for each pure strategy. Assume a player with two information sets h, h' and action sets:

   $$A(h) = \{a, b, c, d\} \text{ and } A(h') = \{1, 2, 3\}$$

   Then there are 11 (i.e. $[A(h) \bigtimes A(h')] - 1$) mixed strategies:

   $$S_i = \{a1, a2, a3, b1, b2, b3, c1, c2, c3, d1, d2\}$$

2. Behavioral strategies:

   $$\beta : h \in H_i \longrightarrow \beta_i(h) \in \Delta(A(h))$$

   For every information set, you choose how to randomize. Then you have 5 possible mixed strategies.

Let's see an example.

Assume for player 1 four possible strategies, with the following probabilities: (Aa),$\sigma_1 =$ .72; (Ab),$\sigma_1 = .08$; (Ba),$\sigma_1 = .18$; (Bb),$\sigma_1 = .02$). Assume also the following behavioral strategies:.

$$A = .8, B = .2, a = .9, a = .1$$

Assume that player 2 plays R. Mixed strategies are: at terminal node z_1: 0.2 (Ba + Bb); at terminal node z_2: (.8) (Aa + Ab); at terminal nodes z_3 and z_4: 0 Behavioral strategies are: 0.2 and 0.8 at z_1 and z_2.

Assume that player 2 plays L. Mixed strategies are: at terminal node z_1: 0.2; at terminal node z_3: (.08); at terminal nodes z_4: (.72), and z_2: 0 Behavioral strategies are: .3, .08 and .72 at z_1, z_3 and z_4.

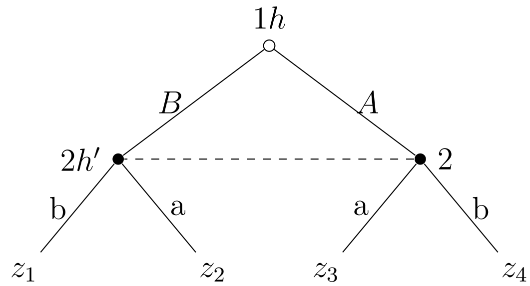
Proposition 9. *A strategy σ_i (mixed or behavioral) is equivalent to another strategy τ_i (mixed or behavioral), if and only if $\forall z \in Z$ and $s_{-i} \in S_{-i}$:*

$$\mathcal{O}(z|\sigma_i, s_{-i}) = \mathcal{O}(z|\tau_i, s_{-i})$$

$\mathcal{O} \in \Delta(Z)$ (where $\mathcal{O}$ indicates the probability of an outcome).

Proof. (Proof is missing) □

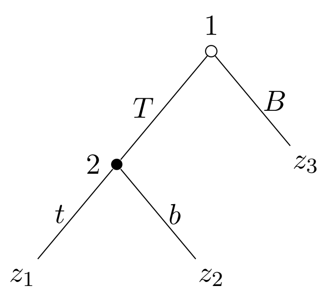
Then, for every behavioral strategy β_i, it exists an equivalent mixed strategy σ_i. Indeed, $\forall s_i,\ s_i : h \in H_i \longrightarrow \mathcal{O}_i \in A(h)$ we can write $\sigma_i(S_i) = \prod_{h \in H_i} \beta_i(s_i(h)|h)$ for all $s_i \in S_i$. The opposite is not always true. Let's see that with an example.



Player 1's strategies are ((Aa),(Ab),(Ba),(Bb)). Let's take mixed strategies $\sigma_1 = (\frac{1}{2}, 0, 0, \frac{1}{2})$. $\beta_i(h) \in \Delta(\{A, B\})$ and $\beta_i(h') \in \Delta(\{a, b\})$. $\beta_1(a|h') > 0$ and $\beta_1(B|h) > 0$. But then z_3 is reached with strictly positive probability. Then it is true that for any mixed strategy, it can be found a behavioral strategy.

Theorem 4.2.1 (Kuhn Theorem)**.** *If a game has perfect recall, for all mixed strategies $\sigma_i \in \Delta(S_i)$, then exists an equivalent behavioral strategy.*

Proof. (Proof is missing) □



We can reach the outcome z_1 with mixed strategy $\sigma_1(Tt)$. The outcome z_2 with $\sigma_1(Tb)$ and z_3 with $\sigma_1(Bt) + \sigma_1(Bb)$.

Take $\sigma_1(Tt) + \sigma_1(Tb) > 0$. Then we can write:

- $\beta_1(t|h') = \frac{\sigma_1(Tt)}{\sigma_1(Tt)+\sigma_1(Tb)}$
- $\beta_1(b|h') = \frac{\sigma_1(Tb)}{\sigma_1(Tt)+\sigma_1(Tt)}$
- $\beta_1(T|h) = \sigma_1(Bt) + \sigma_1(Bb)$

- $\beta_1(B|h) = \sigma_1(Bt) + \sigma_1(Bb)$

Take $\sigma_1(Tt) + \sigma_1(Tb) = 0$. Then we can write:

- $\beta_1(t|h') = x \in [0, 1]$
- $\beta_1(T|h) = 1$
- $\beta_1(B|h) = 0$

Then, from the outcome $\mathcal{O} \in \Delta(Z)$, we can write $u_1(\sigma)$ as $\pi \in \Delta(Z)$ and $u_i(\pi) = \sum_{z \in Z} \pi(z) u_i(z)$.

4.3 Nash Equilibrium

Let's define now NE for games in extensive form

Definition 4.3.1. σ is a Nash Equilibrium if $\forall i$ and $\forall s_i$

$$u_i(\sigma_i^*, \sigma_{-1}^*) \geq u_i(s_1, \sigma_{-i}^*)$$

At the beginning of the game, each player computes the expected utility and looks for it to be optimal. But NE is not a sufficient solution for extensive form games. We can see it through an example.

Example 14: an entry game

Let's recall the entry game seen at the very beginning of this section. An entrant must choose if enter into a market with an incumbent. If it does, a price war arises. The tree is represented in the figure.

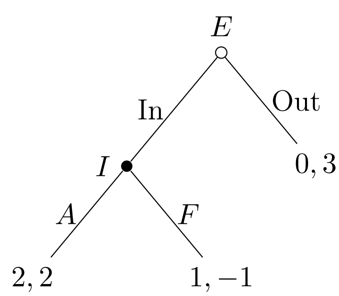


Figure 4.9: An entry game

In this game, there are two Pure Strategies NE: (**Out, F**) and (**In, A**). If the entrant plays out, the incumbent gets a payoff equal to 3, no matter what. Instead, the payoff for the entrant can be written as:

$$
\begin{aligned}
&-1(\sigma_I(F)) + 2(1 - \sigma_I(F)) \geq 0 \\
&\sigma_I(F) \geq \frac{2}{3}.
\end{aligned}
$$

So a Mixed Strategy NE is ((Out),$\sigma_I(F)$), where $\sigma_I(F) \geq \frac{2}{3}$. But this NE is based on a *Non-credible threat.*

4.4 Backward Induction

If games are finite[1] and have perfect information (i.e., games where every information set is a singleton), we can use a technique called *Backward Induction.* Therefore we can find a Backward Induction Solution.

Let's analyze the game in tree form in figure 15.

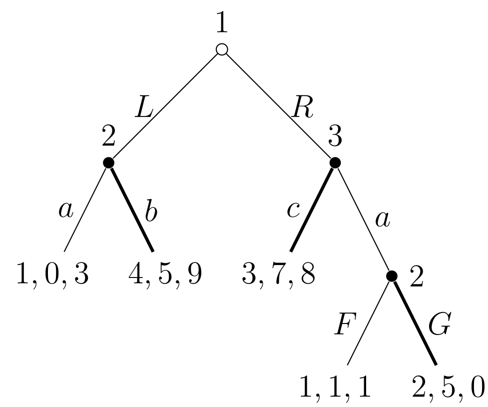


Figure 4.10: Backward Induction Solution

From the graph, we can see that player 1's best strategy is to play **L**, since she can reach her max payoff,4. Player 2's best strategy is to play **bG**. Finally, player 3's best strategy is to play **c**.

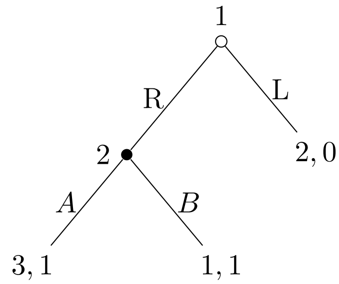
We can outline the two following propositions (without proof).

Proposition 10 (Zermelo-Kuhn Theorem)**.** *Any finite game with perfect information has a backward induction solution in pure strategy*

Proposition 11. *Any Backward Induction Solution is a Nash Equilibrium. Still, the converse is not true.*

Let's now assess, through a simple example, the issue of the uniqueness of the BIS. Take the game tree in the following figure.

[1]To be most precise, the game must have finite horizon. For the differences between games with finite horizon and finite games see above



The Backward Induction Solutions are: (**R,A**),(**L,B**), (**L**,$\sigma_2(A)$),(**R**,$\sigma_2(A)$), where $\sigma_2(A) \geq \frac{1}{2}$ Since $\sigma_1(L) = \sigma_2(A) = \frac{1}{2}$, player 1 randomizes. And since the second player is indifferent, then there are many BIS. *Generically*, BIS is unique in Pure Strategies.

For what concerns the link between BIS and Weakly Dominated Strategies, let's see Figure 16. In the figure below, **L** weakly dominates **R**, but the BIS are (**RB**) and (**LB**). *Generically* BIS is in weakly undominated strategies.

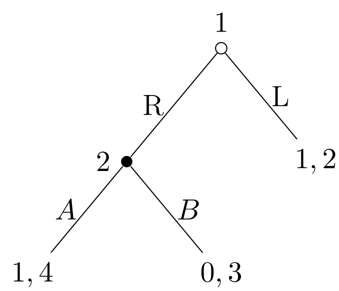


Figure 4.11: BIS and weakly dominated strategies

Example 15: The Centipede Game

Let's now consider the following game. There are two players. Each player must choose between [**S**,**C**]. If **S** is chosen, the game finishes. If **C** is chosen, the game continues. The payoffs grow as the game is long. Furthermore, as the game is played, each player gets the payoff in the following way: the total payoff is doubled, but player i gives 1 unit to player j. The game is finite. This is called the "Centipede Game" because, in the original version, in the final round, each player has the possibility of obtaining a payoff equal to 100. In this game, roughly speaking, we can see two conflicting "attitudes." On the one hand, each player wants to continue since as long as the game is played, the payoff is greater. On the other, continuing, the opponent steals some payoff from her so that each player wants to stop.

Without loss of generality, we can represent this game in the following (shortest) tree form.

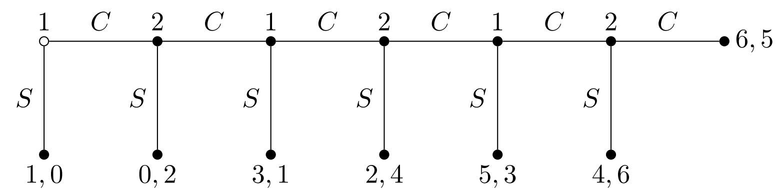


Figure 4.12: A simplified version of the Centipede Game

The BIS for the two players are (**S**,**S**,**S**) and (**S**,**S**,**S**). This is a NE. But there are others. Another NE is that player 1 plays **S** in the first information set, and player 2 plays **S** in his first information set with a probability close to 1 (but less than 1).

4.5 Subgame Perfect Equilibrium

We cannot apply BIS if the game has no perfect information and a finite horizon. A more general concept is that of Subgame Perfect Equilibrium. This idea has been developed mainly by Reinhardt Selten.

In order to define that, we must start defining the idea of subgames and subtrees.

Definition 4.5.1 (Subtree)**.** A subtree consists of one node and all his successors, together with the precedence relation q on these nodes

Definition 4.5.2 (Subgame)**.** A subgame of an extensive form game is an extensive form game whose tree is a subtree of the tree of the original game, and the information set and the payoff are as in the original game

Since it is evident, from the definition above, that even the original game is, in reality, a subgame, then we need a further definition, that of a proper subgame.

Definition 4.5.3 (Proper Subgame)**.** A proper subgame is a subgame that is not the original game

Then we can now define the *Subgame Perfect Equilibrium.*

Definition 4.5.4 (Subgame Perfect Equilibrium)**.** A Subgame Perfect Equilibrium is a profile of behavioral strategies such that their restriction to any subgame is a Nash Equilibrium of that subgame. A Subgame Perfect Equilibrium must be a Nash Equilibrium in the original game and in the subgame.

Example 16: Sequential Bargaining

This example was developed in an important paper by Ariel Rubinstein (1982). Two players must split a dollar. One player makes a proposal $(s_1, 1-s_1)$. If player 2 accepts, then the game stops and each player receives what has been proposed. Otherwise, the game continues, and this time player 2 makes a proposal $(s_2, 1-s_2)$ (notice that $(s_i$ always refers to what player 1 gets). However, then, what each player receives is discounted by $\delta \in (0,1)$. Roughly speaking, the delay is costly.

The game has an infinite horizon. However, we can start with a simplified version, where there are only 3 periods:

- In period 1, player 1 makes a proposal. If it is not accepted, then the game moves to the second period.
- In period 2, player 2 makes a proposal. If it is not accepted, then the game moves to the third (and final) period.
- In period 3 there is a split.

We can represent this situation in the tree in Figure 18. Since this game has finite horizon, we can use backward induction.

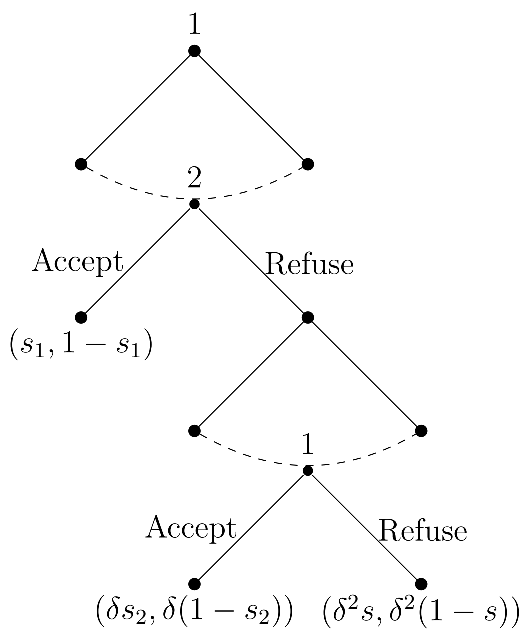


Figure 4.13: Three stages Rubinstein Sequential Bargaining

- In period 3, player 1 receives s_2 if he accepts, and δs if he refuses. Therefore, he accepts if $s_2 \geq \delta s$. It is clear that player 1 accepts if $s_2 > \delta s$. But he does that also if $s_2 = \delta s$. To understand that, let's look at player 2. Take δs as a threshold. If player 2 offers $s_2 < \delta s$, then player 1 will refuse, and player 2 will get $\delta(1-s)$. Let's say player 1 refuses $s_2 = \delta s$ with probability $\beta > 0$. Then player 2 receives $(1-\beta)(1-\delta s) + \beta(1-s_2)$. This is less than $1-\tilde{s}$ where $\tilde{s}$ is close and larger than δs (notice that $1-\delta s > \delta(1-s)$.

- In period 2, at any information set, player 1 accepts s_2 if $s_2 \geq \delta s$. At any information set, player 2 offers $s_2 = \delta s$.

- In period 1, player 1 makes an offer, s_1. If player 2 accepts, he gets $1 - s_1$. If he refuses, he will get $\delta(1 - \delta s)$. Player 2 accepts if and only if:

$$1 - s_1 \geq 1 - \delta(1 - \delta s)$$
$$s_1 \leq 1 - \delta(1 - \delta s)$$

  But then notice that: $1-\delta(1-\delta s) > \delta^2 s$. Therefore at period 1, at any information set, player 2 accepts if and only $s_1 \leq 1 - \delta(1 - \delta s)$. Then, at the initial node, player 1 offers $s_1 = 1 - \delta(1 - \delta s)$

In this version of the game, with perfect horizon, the only Subgame Perfect Equilibrium is: player 1 offers $s_1 = 1 - \delta(1 - \delta s)$ at the initial node, and player 2 accepts.

Let's now focus on the Infinite Horizon Game. In this game, we can write the following payoffs. If players reach an agreement on $(s_j, 1 - s_j)$ in period t, the payoff evaluated in the first period are $\delta^{t-1}s_j, 1 - \delta^{t-1}(1 - s_j)$. If there is no agreement, the payoff is $(0, 0)$. So we can write for player 1:

$$u_1(s_1) = \begin{cases} \delta^{t-1}s_j & \text{if an agreement is reached at period } t \\ 0 & \text{if there is no agreement} \end{cases}$$

And for player 2, we can write:

$$u_2(s_2) = \begin{cases} \delta^{t-1}(1 - s_j) & \text{if an agreement is reached at period } t \\ 0 & \text{if there is no agreement} \end{cases}$$

We can devise two strategies: **Tough**, where a player accepts everything for himself. Therefore he accepts a proposal if and only if he receives everything for himself. **Soft**, where a player offers to the other everything and accepts everything. We can therefore find three possible NE: (**Tough,Soft**),(**Soft,Tough**),(**Tough,Tough**). However, they are not SPE.

Notice that, in the game with infinite horizon, all the subgames are equal. The only difference is that in some subgames, player 1 makes the first offer, and in some others, player 2 makes it. In the subgames where 1 makes the first offer, we can define:

- $\bar{v}_1$ is the max payoff of player 1 across all the SPE

- $\underline{v}_1$ is the min payoff of player 1 across all the SPE

In the subgames where 2 makes the first offer, player 2 can receive at least $1 - \delta\bar{v}_1$. Indeed, player 1 cannot accept an offer greater than $\delta\bar{v}_1$. Equally, she can get at most $1 - \delta\underline{v}_1$. Assume she makes an offer s_2. If this offer is accepted, then she gets $1 - s_2$. But s_2 cannot be lesser than $\delta\underline{v}_1$, therefore $1 - s_2$ must be at most $1 - \delta\underline{v}_1$. Assume the offer s_2 is rejected. In the next period, player 2 will get at most $1 - \underline{v}_1$. But $\delta(1 - \underline{v}_1) < 1 - \delta\underline{v}_1$.

Go to the subgame where 1 makes the first offer. Player 1 can get at least $1 - \delta(1 - \delta\underline{v}_1)$. If player 1 gives to 2 $\delta(1 - \delta\underline{v}_1)$, then player 2 accepts. Then, $\underline{v}_1 \geq 1 - \delta(1 - \delta v_1)$. Player 1 can get, at most, $1 - \delta(1 - \delta\hat{v}_1)$ if the offer is accepted. If the offer is rejected, he can obtain at most $\delta(1 - \delta\hat{v}_1)$. This is because tomorrow he can obtain at most $\delta\underline{v}_1$ but $\delta^2\underline{v}_1 < \delta(1 - \delta\underline{v}_1)$.

Then we have:

$$
\begin{aligned}
\underline{v}_1 &\geq 1 - \delta(1 - \delta\underline{v}_1) \\
\underline{v}_1 &\geq 1 - \delta + \delta^2\underline{v}_1 \\
\underline{v}_1(1 - \delta^2) &\geq 1 - \delta \\
\underline{v}_1 &\geq \frac{1}{1 + \delta}
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{v}_1 &\leq 1 - \delta(1 - \delta\bar{v}_1) \\
\bar{v}_1 &\leq 1 - \delta + \delta^2\bar{v}_1 \\
\bar{v}_1(1 - \delta^2) &\leq 1 - \delta \\
\bar{v}_1 &\leq \frac{1}{1 + \delta}
\end{aligned}
$$

Therefore we have: $\underline{v}_1 = \bar{v}_1 = \frac{1}{1+\delta}$.

Let's see now the players where player 2 makes the first offer. In any subgame perfect equilibrium, player 1's payoff is $\frac{1}{1+\delta}$. The payoff of player 2 cannot be greater than $\frac{\delta}{1+\delta}$, otherwise the sum of the payoffs is greater than 1. At the same time, it cannot be less than $\frac{\delta}{1+\delta}$. Indeed, if he gets less than this, at time t, he will receive anyway $\delta(\frac{1}{1+\delta})$. So player 2's payoff is $\frac{\delta}{1+\delta}$.

This is the (unique) Subgame Perfect Equilibrium of this game: player 1, at any information set, offers $s_1 = \frac{1}{1+\delta}$. Player 2 accepts if $s_1 \geq \frac{\delta}{1+\delta}$. Player 2, at any information set, offers $s_2 = \frac{\delta}{1+\delta}$ and accepts s_1 if and only if $1 - s_1 \geq \frac{\delta}{1+\delta}$, which means $s_1 \leq \frac{1}{1+\delta}$.

In equilibrium, the payoff of the players is $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$, and $\frac{1}{1+\delta} > \frac{\delta}{1+\delta}$.

Notice that in this game, the first mover has an advantage. Indeed it is costly for the other player to reject the first offer. Actually, the numerical value of the payoff depends on the discount factor δ, which measures impatience. If players are very impatient, so that $\delta \sim 0$, then the payoffs are $\sim (1, 0)$. On the contrary, if they are less impatient, say $\delta \sim 1$, then the payoffs are $\sim (\frac{1}{2}.\frac{1}{2})$.

Chapter 5

Repeated Games

Roughly speaking, these are simply games that are repeated several times. In many games, players have the incentive to behave opportunistically, but if the game is repeated several times, possibly infinitely many times, then the players have the incentive to cooperate.

There are three types of repeated games:

- Games with perfect monitoring: these are games where players can observe the moves of others.
- Games with imperfect monitoring: in these games, players do not actually observe what the others do but receive a signal at the end of any stage.
- Games with private monitoring: in these games, the outcome in each period is not directly observable.

In these notes, the focus will be just on games with perfect monitoring.

Let's start with an example, the Battle of Sexes again, but with a slight modification. The new payoff matrix is:

Alice | Bob	Opera	Football	C_2
Opera	3,1	0,0	6,0
Football	0,0	1,3	0,0
C_1	0,0	0,0	5,5

If the game is played just once, the NE are those of the Battle of Sexes: (**Op, Op**), (**F, F**), $\sigma^* = (\frac{1}{4}, \frac{3}{4})$. Indeed, C_1 is strictly dominated by $\alpha(\text{Op}) + (1-\alpha)(F)$ with $\alpha \sim 1$, and C_2 is strictly dominated by $\beta(\text{Op}) + (1-\beta)(\text{Op})$ with $\beta \in (0,1)$.

But suppose that the game is played twice. Player i plays C_i in period 1. Assume further the following strategy: if in period 1, the action profile is (C_1, C_2), in the second period, play **Op**. Otherwise, play **F**.

The total payoff is the sum of the payoffs of each period. Then player 1's payoff, if he follows this strategy, is $5 + 3 = 8$. The payoff if he deviates is $6 + 1 = 7$. So he does not deviate. The same does player 2. If he plays this strategy, he gets $5+1=6$. Otherwise, he gets $0+3=3$. So in the first stage, both players have the incentive to play (C_1, C_2). So we can construct a NE in 2 periods, which is supported by strategy (C_1, C_2) even if this is not a NE in the first-period game.

5.1 Fundamentals

We start defining the stage game, which is a game as usual. $G = (A_1, \ldots, A_n, g_1, \ldots, g_n)$, where A_j is the set of actions, and $A = \bigtimes A_j$. Each player's payoff is given by $g_i : A \longrightarrow \mathbb{R}$. However, now G is played several times, $t = 1, 2, \ldots, T$, where $T \leq \infty$.

Therefore, we define h^t as the history of the game from the beginning until the time t. Taking $a_i \in A_i$, then we can write $h^t = (a_i^0, a_i^1, a_1^2, \ldots, a_i^{t-1})$. Also, we can write $a^t = (a_1^t, \ldots, a_n^t)$. H^t is the set of histories at the beginning of t. $H^0 = \{h^0\}$ is the empty history.

For example, in a Prisoner's Dilemma repeated 4 times, a possible history of the game is $h^4 = ((C, D), (C, C), (D, D), (D, C))$.

Notice that since the players have perfect monitoring, then the history of the game is common knowledge. Furthermore, for each history, we can define a subgame. So H^t is the set of possible subgames. If $T < \infty$, we can define a complete history. If $T = \infty$, then $h^\infty = (a^0, a^1, a^2, \ldots)$.

5.2 Pure and Mixed Strategies

If $T < \infty$ we can write the set of available strategies for player i as $S_i = (s_i^1, s_i^2, \ldots, s_i^T)$. Each strategy s_i^t is therefore:

$$s_i^t : H^t \longrightarrow A_i$$

The same with $T = \infty$.

In the case of mixed strategies, we can write $\beta_i = (\beta_i^1, \beta_i^2, \ldots, \beta_i^T)$.:

$$\beta^t : H^t \longrightarrow \Delta(A_i)$$

Given a strategy profile S_i and a strategy s_i^t, we can describe a game in the following way. At $t = 0$, $a^0 = (S_1^0(h^0) \ldots, S_n^0(h^0))$. At $t = 1$, $a^1 = (S_1^1(a^0), \ldots, S_n^1(a^0))$. And $h^2 = (a^0, a^1)$.

If players randomize, given a strategy profile β_i, $\beta = (\beta_1, \ldots, \beta_n)$ and a strategy β_i^t, we can write:

$$Pr(h^{t+1} | (\beta_1, \ldots, \beta_n)) = Pr(a^0|\beta) Pr(a^1|a^0\beta) \ldots Pr(a^T|a^0 \ldots a^{T-1}\beta)$$
$$\prod_{i=1}^{n} \beta_i^t(a_i^t | a_0 \ldots a_{t-1})$$

We can define the payoffs in a formal way. If $T < \infty$, then:

$$u_i(h^{t+1}) = \frac{1-\delta}{1-\delta^{t+1}} \sum_{t=0}^{T} \delta^t g_i(a^t)$$

Where $\delta \in (0, 1]$. If $T = \infty$, instead:

$$u_i(h^\infty) = (1-\delta) \sum_{t=0}^{\infty} \delta^t g_i(a^t)$$

Where $\delta \in (0, 1)$.

$\frac{1-\delta}{1-\delta^{t+1}}$ and $(1-\delta)$ are a sort of normalization factors, which make it easy to compute the payoff with respect to the actual values of each stage game.[1].

5.3 The set of possible payoffs

Let's take the following 2×2 matrix:

1 | 2	C	D
C	4,4	0,5
D	5,0	1,1

These payoffs can also be represented graphically in figure 19.

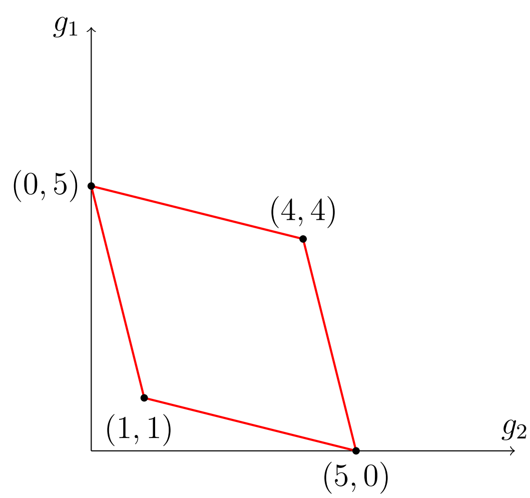


Figure 5.1: The set of possible payoffs of the game above

[1]Notice that they do not affect the value of the payoff functions because we are doing an affine transformation (each utility function is unique up to a positive linear transformation)

This is a Convex Hull of the points representing the payoffs, namely the smallest convex sets containing them. We can write:

$$Co(G) = \Big\{(4,4)(5,0)(0,5)(1,1)\Big\}$$

Any point inside $Co(G)$ can be written as a convex combination of the four points. Therefore we can write, for instance:

$$\frac{1}{3}(4,4) + \frac{1}{3}(5,0) + \frac{1}{3}(0,5)$$

As the expected payoff of playing (C, C),(D, C), and (C, D). Notice that if players play only once they cannot use this randomization (due to the fact that there are independent probability distributions). But this is feasible in repeated games.

Therefore, in the repeated games we can get more than in independent randomization.

Formally, we write the set of all possible payoffs as:

$$V = Co(G) = \Big\{g_1(a), g_2(a), \ldots, g_n(a) | a \in A\Big\}$$

5.4 Solving the Games

We have two different cases: finitely repeated games and infinitely repeated games. Let's start with the earlier.

Finitely Repeated Games

Let's see first the simplest case, that of a game with a unique NE. We take a standard Prisoner's Dilemma repeated finitely many times, i.e. with $T < \infty$ and $\delta \in (0,1]$. We can write the game as $G(T, \delta)$.

1 | 2	C	D
C	4,4	0,5
D	5,0	1,1

Since the game is finite, we can use Backward Induction. At $t = T$, and $i = 1, 2$, $s_i^T(h^T) = D$, for all $h^T \in H^T$. At $t = T-1$, $s_i^{T-1}(h^{T-1} = D$, for all $h^{T-1} \in H^{T-1}$. And so on.

Therefore, in a PD, the unique subgame perfect equilibrium is $\forall i$, $\forall t$, and $\forall h^t \in H^t$, $s_i^t(h^t) = D$. The SPE is unique because the NE of the stage game is unique. This is generalized in the following proposition.

Proposition 12. *Suppose $G(T, \delta)$ has a unique (pure or mixed) Nash Equilibrium, $a^* = (a_1^*, \ldots, a_n^*)$ and $T < \infty$. The $G(T, \delta)$ has an unique Subgame Perfect Equilibrium: $\forall i$, $\forall t$, and $\forall h^t \in H^t$, $s_i^t(h^t) = a^*$*

Let's see a more complicated example, the following 3×3 matrix:

1 | 2	C	D	M
C	4,4	0,5	0,0
D	5,0	1,1	0,0
M	0,0	0,0	3,3

Assume that there are two periods, and for simplicity, $\delta = 1$. In the stage game, there are two pure strategies NE, (D, D), and (M, M). However, we can build a strategy such that a NE is supported by playing (C, C) in the first stage:

$$s_i^t(a^0) = \begin{cases} M & \text{if } a^0 = \text{(C,C)} \\ D & \text{Otherwise} \end{cases}$$

Therefore, if in the first period, it is played (C, C), then in the second, it is played (M, M), a NE. If in the first period, it is not played (C, C), then it is played (D, D), another NE. Still, we can see that if players stay committed to this strategy, they have a greater payoff. Player 1 gets $4 + 3 = 7$, and if he deviates, he receives $5 + 1 = 6$. The same for player 2.

Infinitely Repeated Games

Take a game $G(\infty, \delta)$, where $\delta \in (0, 1)$ (usually it is assumed that δ is close to 1, i.e., that players care about the future). Let's see a Prisoner's Dilemma again and if we can construct an equilibrium strategy sustained by choosing C in the first period (namely, by players cooperating).

1 | 2	C	D
C	4,4	0,5
D	5,0	1,1

Let's write the following strategy (which is called Trigger Strategy):

$$s_i^t(h^t) = \begin{cases} C & \text{if } h^t = \text{((C,C),}\ldots\text{,(C,C))} \\ D & \text{Otherwise} \end{cases}$$

In words, each player plays C if the other plays C. Otherwise, if one plays D, the other player will play D forever. Of course, if player i plays C, the other player will always play C. Take a history h^t = ((C,C),...,(C,C)). Following this strategy, the payoff of the player i is:

$$(1 - \delta)(4 + 4\delta + 4\delta^2 + \dots) \approx 4$$

If a player deviates, the payoff is:

$$\begin{aligned}
&(1-\delta)(5+\delta+\delta^2+\dots) = \\
&(1-\delta)5+\delta = \\
&5-5\delta+\delta = \\
&5-4\delta
\end{aligned}$$

The deviation is profitable only if $5-4\delta > 4$, which means if $\delta \geq \frac{1}{4}$. This means that if players are patient, they can sustain many strategies.

We can generalize this in an important result of Game Theory, known as the Folk Theorem, formally stated, for the first time, by James Friedman in 1971.[2]

Theorem 5.4.1 (Friedman's Folk Theorem)**.** *Let G be a normal form game, and let $(a_1, \dots, a_n)$ be the payoff vector at a Nash Equilibrium of G. Let $(x_1, \dots, x_n) \in V$ denote any other feasible payoff vector. If $x_i > e, \ \ \forall i$, then it exists a $\bar{\delta} < 1$ such that $\forall \delta > \bar{\delta}$, $G(\infty, \delta)$ admits a Subgame Perfect Equilibrium with payoffs $(x_1, \dots, x_n)$.*

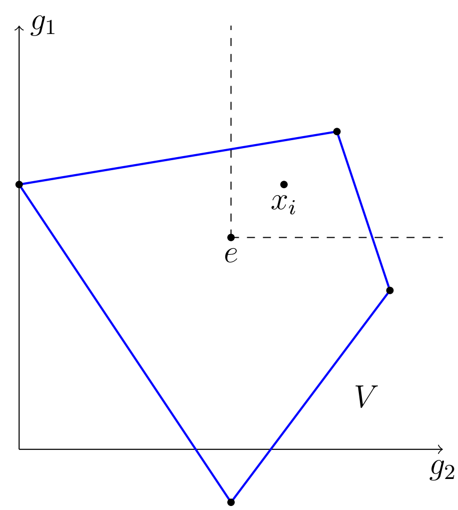


Figure 5.2: Two players graphical representation of the Folk Theorem

Proof. Assume for simplicity that $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n) \ \ \forall i$ and that $x_i = g_i(\bar{a})$. Let $a^* = (a_1^*, \dots, a_n^*)$ be a Nash Equilibrium of G so that $g_i(a) = e$.

Construct the following strategy. $\forall i$, let $s_i(h^0) = \bar{a}_i$ at $t = 0$. At $t > 0$, instead:

$$s_i^t(h^t) = \begin{cases} \bar{a}_i & h^t = (\bar{a} \dots, \bar{a}) \\ a_i^* & \text{otherwise} \end{cases}$$

[2]The name, Folk Theorem, refers to the fact that for some years, in the 1950s and 1960s, the implications of the theorem were common knowledge among the game theorists, even if none had formally stated them

If the players follow this strategy, they choose $\bar{a}_i$ in any period, and they get x in each period. We want to show that this is a Subgame Perfect Equilibrium.

Assume $h^t \neq (\bar{a}, \dots, \bar{a})$. Player j plays a_j^* forever, so player i maximizes his payoff today, playing her NE.

Assume $h^0 = (\bar{a}, \dots, \bar{a})$. If all players follow the strategy, each player's payoff is:

$$(1-\delta)(x_i + \delta x_i + \delta^2 x_i + \dots)$$

If player j deviates, he gets:

$$\begin{aligned}&(1-\delta)(d_i + \delta e + \delta^2 e + \dots) = \\ &(1-\delta)d_i + \delta e\end{aligned}$$

Where $d_i = \max_a g_i(a)$. The deviation is not profitable if $x_i \geq (1-\delta)d_i + \delta e$. This in turns implies $\delta > \frac{d_i - x_i}{d_i - e}$ which is lesser than 1, since $x_i > e$. If we take $\bar{\delta} = \max_{i=1,\dots,n} \frac{d_i - x_i}{d_i - e}$, then we can find a $\delta > \bar{\delta}$ such that s_i^t is a Subgame Perfect Equilibrium, yielding payoffs $x_i > a_i^*$. □

So far, we have seen only the trigger strategy. Let's introduce now another concept, that of MinMax.

Take the stage game $G = (A_i, \dots, A_n, g_1, \dots, g_n)$ and strategy profile of the opponent, a_{-i}, define $w_i(a_{-i} = \max_{a_i \in A_i} g_i(a_i, a_{-i})$ the max payoff i receives when the opponent plays a_{-i}.

Suppose a_{-i} is chosen to minimize w_i, that is, $\underline{v}_i = \min_{a_i} w_i(a_{-i})$. Then $\underline{v}_i$ is the MinMax of player i, the minimum player i gets when playing her best response.

Let's see one example. Take the following matrix.

1 | 2	C	D
C	4,4	0,5
D	5,0	1,1

Player 1 plays D, as well as player 2. The NE is (D, D). Looking at the payoffs, we see that for each player, 1 is the maximum they can obtain, choosing for each strategy the minimum payoff. That is, for player 1, playing C, the minimum is 0. Playing D, the minimum is 1. And $1 > 0$. Notice that $\underline{v}_1 = \underline{v}_2$.

Then we can state and prove the following result.

Proposition 13. *In any Nash Equilibrium of $G(T, \delta)$ the payoff of player i is at least $\underline{v}_i$*

Proof. Fix strategy of opponent β_{-i}. $\forall j, \forall h^* t, \beta(h^t) \in \Delta(A_j)$. Suppose (β_i, β_{-i}) is a Nash Equilibrium, so that $u(\beta_i, \beta_{-i}) \geq u(s_i, \beta_{-i}) \quad \forall s_i$. Take $\tilde{s}_i$. At any history h^t, $\tilde{s}_i^t(h^t) = \arg\max_{a_i} g_i(a_i, \beta_j^t(h^t)_{j \neq i})$. Choose an action to maximize today's payoff that is, $w_i(\beta_j^t(h^t))_{j \neq i} \geq \underline{v}_i$. Therefore, in a Nash Equilibrium, the payoff is larger or equal to it, or w_i is a profitable deviation. □

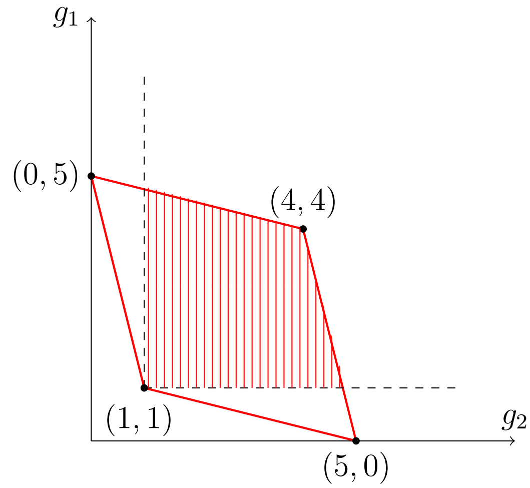


Figure 5.3: Prisoner's Dilemma

Let's see two graphical examples. First the Prisoner's Dilemma (Figure 21). The NE of this game is given by the strategy (D, D), and the payoff is $(1, 1)$. This is also the MinMax for each player. The support set passes from the NE.

A different situation is captured in Figure 22. There the NE is sustained by the support set passing from the MinMax.

Let's now define the concept of One-shot deviations.

Definition 5.4.1. Given a strategy s_i of $G(t, \delta)$, a one-shot deviation by player i at history h^t is a strategy s'_i such that $s'^t_i(\hat{h}^t \neq s^t_i(\hat{h}^t)$ but $s'^t_i(h^t) = s^t_i(h^t)\ \ \forall h^t$ that follows $\hat{h}^t$.

The trigger strategies are an example:

$$s_i^0(h^0) = C$$

$$s_i^t(h^t) = \begin{cases} C & \text{if } h^T = ((C,C), \dots, (C,C)) \\ D & \text{Otherwise} \end{cases}$$

whereas $T = 1, \dots, t-1$. Playing always C is the equilibrium strategy. Playing $(D, C, C, \dots)$ is not a one-shot deviation. Playing always D is a one-shot deviation.

Then, we have the following proposition.

Proposition 14. *Let G be finite, a strategy profile $S = (s_1, \dots, s_n)$ is a Subgame Perfect Equilibrium of $G(\infty, \delta)$ if and only if one-shot deviations are not profitable $\forall i, \forall t, \forall h^t \in H^t$ and for all possible deviations $\tilde{s}_i$ form s_i at h^t. I.e.*

$$u_i(s_i, s_{-i}|h^t) \geq u_i(\tilde{s}_i, s_{-i}|h^t)$$

This result holds also for $G(T, \delta)$ where $T < \infty$

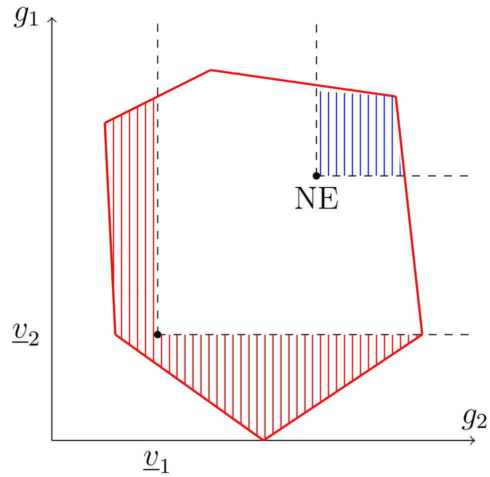


Figure 5.4: MinMax $\neq$ NE

Proof. ($\Rightarrow$) No deviations are profitable by the definition of Subgame Perfect Equilibrium. Therefore, even one-shot deviations are not profitable. ($\Leftarrow$) Let's proceed by contradiction. Assume s is a strategy profile of G and i has a profitable deviation $\hat{s}_i$ at $\tilde{h}_t$. Namely $u_i(\hat{s}_i, s_{-i}|\tilde{h}^t) - u_i(s_i, s_{-i}|\tilde{h}^t) = A > 0$. Choose now $\tilde{t} > t$. Let $\hat{s}_i$ be the strategy:

$$\bar{s}_i(h^t) = \begin{cases} \hat{s}_i(h^t) & \text{If } t < \tilde{t} \\ s_i(h^t) & t \geq \tilde{t} \end{cases}$$

Consider now the difference $|u_i(\bar{s}_i, s_{-i}|\tilde{h}^t) - u_i(\hat{s}_i, s_{-i}|\hat{h}^t)| = (1-\delta)|\sum_t^\infty \delta^t(g_i(a_t|\bar{s}_i, s_{-i}, \tilde{h}^t) - g_i(a^t|\hat{s}_i, s_{-i}, \tilde{h}^t))| \leq (1-\delta)\sum_t^\infty \delta^t |g_i(\bar{s}_i, s_{-i}, \tilde{h}^t) - g_i(a^t|s_i^t, s_{-i}, \tilde{h}^t))|$.[3]

Moreover, we have $(-\delta)\sum_t^\infty \delta^t |g_i(a^t|\bar{s}_i, s_{-i}, \tilde{h}^t) - g_i(a^t|\hat{s}_i, s_{-i}, \tilde{h}))| \leq (1-\delta)\sum_t^\infty \delta^t M = \delta^t M$ for some $M > 0$, since G has bounded payoffs. Let $t \to \infty$ so $\delta^t M \to 0$. Hence, if t is large enough, then $\bar{s}$ is also a profitable deviation from s_i at h^t. Then unilateral deviations are not profitable.

It remains to prove that s is a Subgame Perfect Equilibrium. That is, we must show that no deviation is profitable. Starting at h^t, assume that the histories where s_i and $\bar{s}_i$ are different is N. Since G is finite and one-shot deviations are never profitable, then $1 < N < \infty$. Since s_i and $\hat{s}_i$ differ at N histories, then it exists a h^t that follows h^{t0} such that $\hat{s}_i(h^t) \neq \hat{s}_j(h^T)\ \ \forall T > t$. Then $\hat{s}_i$ is a one-shot deviation from s_i at h^t. However, one-shot deviations are never profitable, so $u_i(s_i, s_{-i}|h^t) \geq u_i(\hat{s}_i, s_{-i}|h^t)$. Constructing an $\tilde{s}_i$ equal to $\hat{s}_i$, then at h^t, we get $\tilde{s}_i(h^t) = s_i(h^t)$. Then, if $\hat{s}_i$ is a profitable deviation from s_i at h^{t0} so it is $\tilde{s}_i(h^{t0})$. Therefore, $\tilde{s}_i$ is now the profitable deviation with respect to s_i that differs from it $(N-1)$ times, reaching a contradiction. Hence, if a profitable deviation from the strategy profile s exists, we conclude that s is not a Subgame Perfect Equilibrium.

[3]This holds due to $|A + B| \leq |A| + |B|$. This is known as the Triangle Inequality

$\square$

Let's see an example. Take the Prisoner's Dilemma matrix and define the following strategy.

1 | 2	C	D
C	4,4	0,5
D	5,0	1,1

$$s_i^0(h^0) = C \quad \text{and} \quad s_i^t(h^t) = \begin{cases} C & \text{if } a_j^{t-1} = C \\ D & \text{if } a_j^{t-1} = D \end{cases}$$

This strategy is called "Tit for Tat." It states that in the first period, a player plays **C**. After, he plays the strategy of her opponent on the period before. This strategy implies that every player has a memory of what happened in the period before. We can classify the different histories of the game as $H_{CC}, H_{CD}, H_{DC}, H_{DD}$, where $h^t \in H_{a_1,a_2}$ and $a^{t-1} = (a_1, a_2)$. We can also define the continuation payoff for player i at $h^t \in H_{a_1,a_2}$ as W_{a_1,a_2}. Then we can write:

$$W_{CC} = 4 + 4\delta + 4\delta^2 + 4\delta^3 + \cdots = \frac{4}{1-\delta}$$

$$W_{CD} = 5 + 0\delta + 5\delta^2 + 0\delta^3 + \cdots = \frac{5}{1-\delta^2}$$

$$W_{DC} = 0 + 5\delta + 0\delta^2 + 5\delta^3 + \cdots = \frac{5\delta}{1-\delta^2}$$

$$W_{DD} = 1 + \delta + \delta^2 + \delta^3 + \cdots = \frac{1}{1-\delta}$$

If one player deviates in one period, she is punished in the following periods. So that, to have an equilibrium, it must be the case that, for player i, the continuation payoff associated with each different history be greater or equal to the possible deviations. Namely, the continuation payoff for the strategy "both cooperate" be greater or equal to the payoff player i receives playing **D** plus the discounted value of the continuation payoff W_{DC}. That is:

$$
\begin{aligned}
&W_{CC} \geq 5 + \delta W_{DC} = \\
&\frac{4}{1-\delta} \geq 5 + \delta(\frac{5\delta}{1-\delta^2}) \\
&\frac{4}{1-\delta} \geq \frac{5(1-\delta^2)+5\delta^2}{1-\delta^2} \\
&\frac{4}{1-\delta} \geq \frac{5-5\delta^2+5\delta^2}{1-\delta^2} \\
&\frac{4}{1-\delta} - \frac{5}{1-\delta^2} \geq 0 \\
&\frac{4}{1-\delta} + \frac{5}{(1-\delta)(1+\delta)} \\
&\frac{4-4\delta-5}{(1-\delta)(1+\delta)} \geq 0 \\
&\frac{4\delta-1}{1-\delta^2} \geq 0 \\
&\delta \geq \frac{1}{4}
\end{aligned}
$$

This is for all continuation payoffs. Then solving each inequality:

$$W_{CD} \geq 4 + \delta W_{CC} \Rightarrow \delta \leq \frac{1}{4}$$

$$W_{DC} \geq 1 + \delta W_{DD} \Rightarrow \delta \geq \frac{1}{4}$$

$$W_{DD} \geq 0 + \delta W_{CD} \Rightarrow \delta \leq \frac{1}{4}$$

Therefore, we have $\delta = \frac{1}{4}$. Tit for Tat is a Subgame Perfect Equilibrium if $\delta = \frac{1}{4}$.

A general version of the Folk Theorem has been proved in 1986 by Drew Fudenberg and Eric Maskin.

Theorem 5.4.2 (Fudenberg & Maskin's Folk Theorem)**.** *Assume one of the following: (i) $n = 2$ (ii) V has a non-empty interior.*

Then, for all $v \in V$ with $v_i \geq \underline{v}_i$ $\forall i$, it exists a $\bar{\delta} < 1$ such that $\delta \geq \bar{\delta}$ and it exists a Subgame Perfect Equilibrium σ of $G(\infty, \delta)$ such that $u_i(\sigma) = v_i$ $\forall i$.

Proof. Assume $a = (a_1, \ldots, a_n)$ such that $\forall i$ $g_i(a) = v_i$. Assume also that α_j MinMax strategies of j's opponent, such that $g_j(\alpha_j^*, \alpha_{-j}) = \underline{v}_j$ $\forall j$ and $g_i(\alpha_j^*, \alpha_{-j}) = w_i^j$ $\forall i \neq j$. Because, from (i), V has a non-empty interior, it exists a $v' = (v_1', v_2', \ldots, v_n')$ and $\epsilon > 0$ such that $\forall i$ $\underline{v}_i < v_i' < v_i$ and $\forall i$ $v^i(i) = (v_1' + \epsilon \ldots, v_n' + \epsilon) \in V$. Let's also assume for simplicity that $\forall i$ it exists an action profile $a(i)$ such that, $\forall j$ $g_j(a(i)) = v_j'(i)$.

Let's try to construct a strategy profile that is a Subgame Perfect Equilibrium for δ big enough. The game has different phases.

In phase 1, every player plays strategy a to obtain a payoff v. The game remains in phase 1 unless there is a unilateral deviation from a. If there is a deviation by player j, the game moves to phase 2_j.

In phase 2_j, players play $(\alpha_j^*, \alpha_{-j})$ for N-periods, unless there is unilateral deviation from $(\alpha_j^*, \alpha_{-j})$. After N-periods without unilateral deviation, the game moves to phase 3_j. Otherwise, the game moves to phase 2_i for deviations of player i (i is the index of the deviating player, it can be also player j).

In phase 3_j, players play $a(j)$, to obtain payoff $v'(j)$. The game remains here unless there is a unilateral deviation by a player i (including also j). In this case, the game is moved to 2_i.

In phase 1 player i gets v_i. One shot deviation gets at most $(1-\delta)\max_a g_i(a) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i$. This is smaller than v_i for δ large enough, so there is no incentive to deviate for player i.

In phase 3_j, player i gets $v_i' + \epsilon$. One-shot deviation gets at most $(1-\delta)\max_a g_i(a) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i'$. Again $v_i' + \epsilon > v_i'$, so there is no incentive to deviate.

In phase 3_i, player i gets v_i'. One shot deviation gets at most $(1-\delta)\max_a g_i(a) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i'$, which is less than v_i' for δ large enough.

Let's now consider phase 2_j with $j \neq i$ and $N' \leq N$ left periods. Following the equilibrium strategy, player i gets $(1-\delta^N)w_i^j + \delta^{N'}(v_i + \epsilon)$. One shot deviation gets at most $(1-\delta)\max_a g_i(a) + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i'$. As $\delta \to 1$, this goes to v_i', so there is no incentive toward unilateral deviation when δ is large enough.

Finally, consider phase 2_j, con $N' < N$ periods left. Player i gets $(1+\delta^{N'})\underline{v}_i + \delta^{N'}v_i$. One shot-deviations get at most $(1-\delta)\underline{v}_i + \delta(1-\delta^N)\underline{v}_i + \delta^{N+1}v_i$ which is equal to $(1-\delta^{N+1})\underline{v}_i + \delta^{N+1}v_i'$. But $(1+\delta^{N'})\underline{v}_i + \delta^{N'}v_i < (1-\delta^{N+1})\underline{v}_i + \delta^{N+1}v_i'$, since $(N' < N)$. Then, there is no incentive to deviate.

This concludes the proof since the same reasoning can be extended to phase 1 and to all players. □

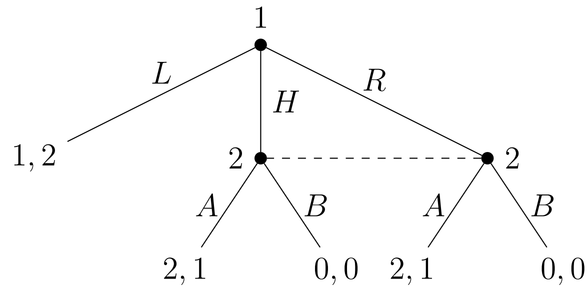
5.5 Stronger notions of equilibria

As far as the games become richer and more complicated, stronger notions of equilibrium are required to rule out implausible equilibria. This is what we need for Bayesian Games in extensive form. In particular, as we have seen, some equilibria embody non-credible strategies as non-credible treats.

Let's see the following tree:

(**L**,**B**) is a Nash Equilibrium. However, **B** is not a credible threat since if player 2 plays it, she gets 0 instead of 2. Furthermore, **B** is strictly dominated. However, it is not enough to rule out strictly dominated strategies. Notice also that the tree above does not have proper subgames.

See now the following example, represented in Figure 23. This is called Selten's Horse. A NE and SPE of this game is ((**Du**),**a**,**L**). But notice that strategy **a** is not credible since if player 2 plays it, she gets less than she would receive otherwise. Another



way of putting this is that at her information node, 2 will never play **a**.

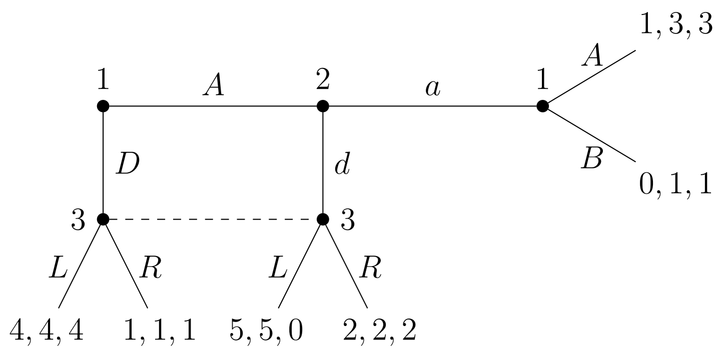


Figure 5.5: Selten's Horse

Further refinement is needed. This is the idea of a system of beliefs.

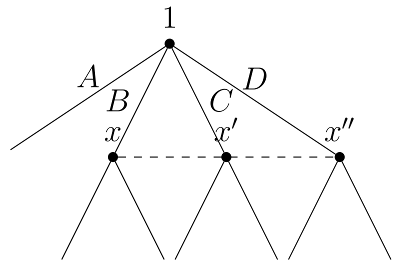
Definition 5.5.1 (System of Beliefs)**.** Given $\Gamma = (X, Z, q, N, \phi, A, H, p, u_1, \ldots, u_n)$, an extensive form game, a system of beliefs is a mapping $\mu : X \to [0, 1]$, such that $\forall i, \forall h \in H_i$, $\sum_{x \in h} \mu(x) = 1$

In other words, a system of beliefs is the probability of being in each node given the fact that a player is in a certain info set. This can be computed in the following way. Having a strategy $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\mu(x) = \frac{Pr(x|\sigma)}{\sum_{x' \in h} Pr(x'|\sigma)}$ (supposing that $\sum_{x' \in h} Pr(x'|\sigma) > 0$ If an information set h is reached with positive probability, we say that h is on-path. Otherwise, that information set is said to be off-path.

A strategy profile σ together with a system of beliefs μ define an assessment. Given this, we can define the notion of Sequential Rationality.

Definition 5.5.2 (Sequential Rationality)**.** An Assessment (σ, μ) is sequentially rational if, $\forall i, \forall h \in H_i$, then $\sum_{x \in h} \mu(x) u_i(\sigma_i, \sigma_{-i}|x) \geq \sum_{x \in h} \mu(x) u_i(s_i, \sigma_{-i}|x) \quad \forall i$

If an information set is feasible (that is, if it is reached when it is played strategy σ), then you can reach it with probability ≤ 1. See the following example.



Since $\beta_1(A) < 1$, then we can write, following the Bayes' rule:

$$\mu(x) = \frac{\beta_1(B)}{\underbrace{\beta_1(B) + \beta_1(C) + \beta_1(D)}_{\equiv 1-\beta_1(A)}}$$

Let's see another example, represented in figure 24. Look only to the probabilities of player 2. Then $\mu(x) = \frac{3}{4}$ and $\mu(x') = \frac{1}{4}$. If player 4 plays **D**, is **AL** Sequentially Rational?

Then we have $\mu(x)u_2(AL, \sigma_{-2}|x) + \mu(x')u_2(AL, \sigma_{-2}|x') = \frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 0 = \frac{9}{4}$

- What if I deviate to **AR**? $\frac{3}{4} \cdot 3 + \frac{1}{4} \cdot 0 = \frac{9}{4}$.
- What if I deviate to **BL**? $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 2 = \frac{1}{2}$
- What if I deviate to **BR**? $\frac{3}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}$

Then we can conclude that **AL** is Sequentially Rational.

5.5.1 Perfect Bayesian Equilibrium

However, not all systems of beliefs have sense, given certain strategies. An example can be seen in the tree represented in Figure 25. The unique SPE is (**B,b,R**). Suppose (**T,b,L**) with $\mu(x) = 1, \mu(x') = 0$. This strategy is sequentially rational, however beliefs do not make any sense.

We need to impose restrictions on Assessment, in particular to system of beliefs.

This leads to the following definition.

Definition 5.5.3 (Perfect Bayesian Equilibrium)**.** An assessment (σ, μ) is a Perfect Bayesian Equilibrium if it is sequentially rational and the beliefs are derived from strategy σ using Bayes' rule "whenever possible."

Notice that:

- For sure, you can use Bayes' rule when the information set is on path
- Otherwise, it depends on the game.

We can now assess the notion of Sequential Equilibrium.

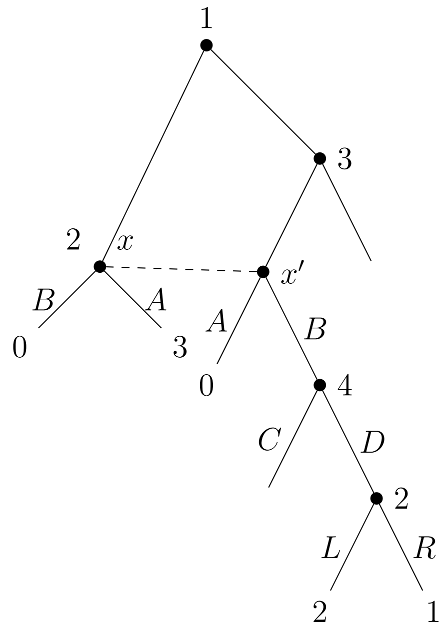


Figure 5.6: Sequential Rationality

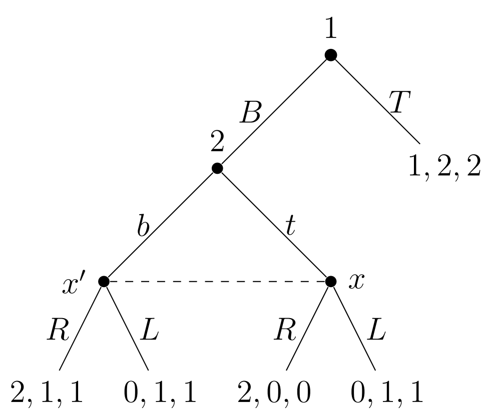


Figure 5.7:

5.5.2 Sequential Equilibrium

To use Bayes'Rule, unless all information sets are on path, you must divide by zero. A way to avoid that is to "perturb" the information set. Let's see the following example.

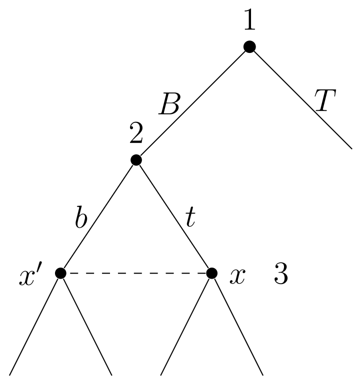


Figure 5.8:

Assume $\beta_1(B) = 1$, $\beta_1(T) = 0$ and $\beta_2(b) = 1$,$\beta_2(t) = 0$. Introducing a perturbation, we have $\beta_1^n(T) = 1 - \epsilon_1^n$, $\beta_1^n(B) = \epsilon_1^n$, $\beta_2^n(t) = \epsilon_2^n$, $\beta_2^n(b) = 1 - \epsilon_2^n$.

Then we can write:

$$\mu^n(x) = \frac{\epsilon_1^n \cdot (1 - \epsilon_2^n)}{\epsilon_1^n \cdot (1 - \epsilon_2^n) + \epsilon_1^n + \epsilon_2^n} = 1 - \epsilon_1^n$$

Therefore, taking the limit, we have $\lim_{n \to \infty} \mu^n(x) = 1$

In general, assuming the tree in Figure 27. If player 1 plays **A**, then the information sets of players 2 and 3 are reached with probabilities 0.

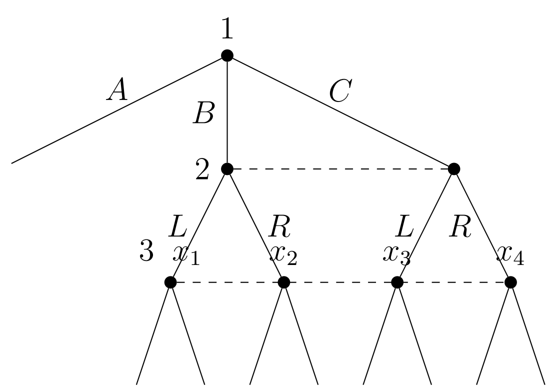


Figure 5.9:

But if $\beta_1^n(B) > 0$ and $\beta_1^n(C) > 0$, and $\beta_2^n(L) > 0$ and $\beta_2^n(R) > 0$, then we can write

$\mu^n(x_1)$ as:

$$\mu^n(x_1) = \frac{\beta_1^n(B)\cdot\beta_2^n(L)}{\beta_1^n(B)\cdot\beta_2^n(L)+\beta_1^n(B)\cdot\beta_2^n(R)+\beta_1^n(C)\cdot\beta_2^n(L)+\beta_1^n(C)\cdot\beta_2^n(R)}$$

And consequently:

$$\mu^n(x_2) = \frac{\beta_1^n(B)\cdot\beta_2^n(R)}{\beta_1^n(B)\cdot\beta_2^n(L)+\beta_1^n(B)\cdot\beta_2^n(R)+\beta_1^n(C)\cdot\beta_2^n(L)+\beta_1^n(C)\cdot\beta_2^n(R)}$$

$$\mu^n(x_3) = \frac{\beta_1^n(C)\cdot\beta_2^n(L)}{\beta_1^n(B)\cdot\beta_2^n(L)+\beta_1^n(B)\cdot\beta_2^n(R)+\beta_1^n(C)\cdot\beta_2^n(L)+\beta_1^n(C)\cdot\beta_2^n(R)}$$

$$\mu^n(x_4) = \frac{\beta_1^n(C)\cdot\beta_2^n(R)}{\beta_1^n(B)\cdot\beta_2^n(L)+\beta_1^n(B)\cdot\beta_2^n(R)+\beta_1^n(C)\cdot\beta_2^n(L)+\beta_1^n(C)\cdot\beta_2^n(R)}$$

And:

$$\underbrace{\mu^n(x_1)\cdot\mu^n(x_4) = \mu^n(x_2)\cdot\mu^n(x_3)}_{\equiv \frac{\mu^n(x_1)}{\mu^n(x_2)} = \frac{\mu^n(x_3)}{\mu^n(x_4)}}$$

A further definition is needed.

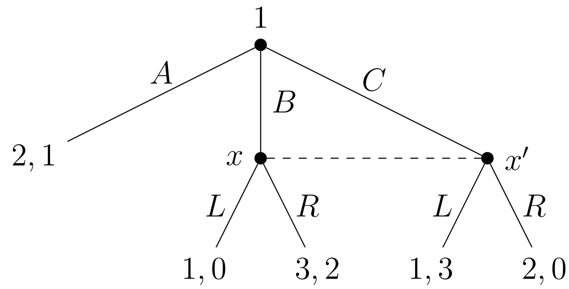
Definition 5.5.4 (Consistency)**.** An Assessment (σ, μ) is consistent if it exists a sequence $\{\sigma^n\}_{n=1}^n$ of completely mixed strategy profiles such that:
i) $\lim_{n\to\infty}\sigma^n = \sigma$
ii)$\lim_{n\to\infty}\mu^n = \mu$
Where $\mu^n(\sigma^n)$ is the profile system of beliefs derived from σ^n using Bayes' Rule.

Then we can define the notion of Sequential Equilibrium (first advanced in Kreps & Wilson 1982)

Definition 5.5.5 (Sequential Equilibrium)**.** An Assessment (σ, μ) is a Sequential Equilibrium if it is:

- Sequentially Rational
- Consistent

Therefore, to find Sequential Equilibria, one has to check for Consistency and Sequential Rationality. The latter condition is satisfied if and only if one-shot deviations are not profitable.



Example 17: Sequential Equilibrium

Let's see the Pure-Strategy Sequential Equilibria. If player 1 plays **B**, then $\mu(x) = 1$. Player 2 plays **R**. A sequential equilibrium is $(\mathbf{B}, \mathbf{R}, \mu(x) = 1)$.

Player 1 plays **C**, then $\mu(x) = 0$ and $\mu(x') = 1$. Player 2 plays **L**, but this is not optimal for player 1. Therefore this is not a sequential equilibrium.

Let's construct a sequential equilibrium where player 1 plays **A**, player 2 plays **L**. **L** is optimal if $3(1 - \mu(x)) \geq 2\mu(x)$ (notice that $\mu(x') = 1 - \mu(x)$). Then $\mu(x) \leq \frac{3}{5}$.

If $\mu(x) = 0$ and $\mu(x') = 1$, then (σ, μ) is sequentially rational. Indeed, take $\mu(x) = \alpha \in (0, \frac{3}{5}]$. Take $\beta_1^n(B) = \alpha\epsilon_n$ and $\beta_1^n(C) = \epsilon_n(1 - \alpha)$. Then $\mu^n(x) = \frac{\epsilon_n\alpha}{\epsilon_n\alpha + \epsilon_n(1-\alpha)} = \alpha$

Let's check for consistency. Define $\beta_1^n(B) = \epsilon_n^2$ and $\beta_1^n(C) = \epsilon_n$ and $\{\epsilon_n\} \to 0$. $\mu^n(x) = \frac{\epsilon_n}{\epsilon_n + \epsilon_n^2} = \frac{1}{1+\epsilon_n} \to 1$

Therefore (σ, μ) is a Sequential Equilibrium.

Another Sequential Equilibrium is: player 1 plays **A**, and player 2 randomizes between **L** and **R**, with $\beta_2(C) \in (0, 1)$ and $\beta_2(R) = 1 - \beta_2(L)$. Only when $\mu(x) = \frac{3}{5}$ the player 2 is indifferent. Then $\beta_1^n(B) = \frac{3}{5}\epsilon^n$, $\beta_1^n(C) = \frac{2}{5}\epsilon_n$. Then $\mu^n(x) = \frac{\frac{3}{5}\epsilon^n}{\frac{3}{5}\epsilon^n + \frac{2}{5}\epsilon^n} = \frac{3}{5}$. For player 2, only randomization is optimal.

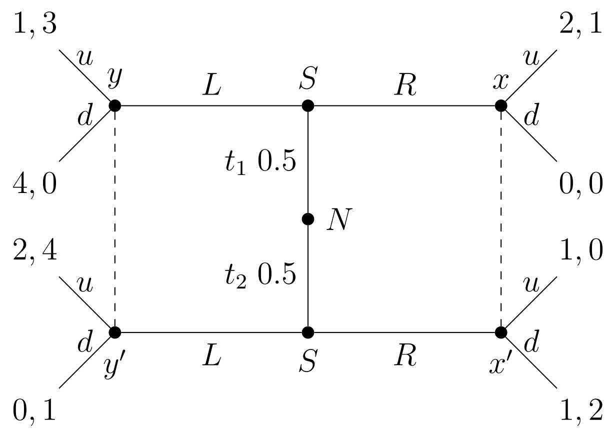
Suppose player 1 plays **A**, the payoff is 2. Expected payoff of playing **B** is $\beta_2(L) + 3(1 - \beta_2(L)$, therefore $\beta_2(L) \geq \frac{1}{2}$

There are no other Sequential Equilibria. Furthermore, if $beta_1(B)$ and $\beta_2(C)$ are both greater than 0, they must be optimal, so player 1 must be indifferent (the case when player 2 plays **L** with probability 1, but then playing **C** is not optimal). Randomizing between **A** and **B**, between **A** and **C**, is not optimal.

Example 18: Signaling Games

These are games between a sender, who is privately informed, and a receiver. The sender (type t) chooses an action a_s. The receiver observes the action a_s and not the type and chooses an action a_r. The payoff depends on a_s, a_r, t. The action signals your type. One of the most famous examples is Spence's Education Model. The ability of each student is known only to her but affects the education he receives. A hiring employee does not know the ability of the student, but he can develop an idea by looking at her education.

An equilibrium can be:



- Fully revealing (separating), when t_1, t_2 choose different actions;
- Pooling, when t_1, t_2 choose the same action

With more than 2 types, we can have partially revealing and partially pooling equilibria.

In any Sequential Equilibrium, t_2 plays **L**. To construct a Separating SE where t_2 plays **L** and t_1 plays **R**. For the receiver **u** strictly dominates **d** ($3 > 0$ and $1 > 0$). **R** is optimal for player S.

Let's now construct a Pooling SE, where t_1 and t_2 play **L**. Take $\mu(y) = \mu(y') = 0.5$. The receiver, when the sender plays L, plays **u**. When the sender plays **R**, the receiver plays **d**, otherwise, the sender plays **L**. The Expected payoff for the receiver is $2 - 2\mu(x) \geq \mu(x)$, then $\mu(x) \leq \frac{2}{3}$.

Subsets of these games are the Cheap-Talk games.

Cheap-Talk Games

In these games, the sender chooses an action, i.e. a message (M). The receiver observes the message but not the type and chooses a_r. The payoff still depends on a_s, a_r, t and not M. Therefore, $\forall a_r, t$ we have $u_i(a_r, t, M) = u_i(a_r, t, M')$. Since the message does not affect the payoffs, these games are called Cheap Talks. But the message, of course, can influence the action of the receiver.

Furthermore, $p \in \Delta(T)$ (where T is the set of sender's types), $p(t)$ is the probability of type t, the receiver's best response, $a_r^* = \arg\max_{a_r} \sum_t p(t)u(a_r, t)$.

Suppose that after every $m \in M$ (where M is the set of messages), the receiver plays a_r^*. The sender does not care about the messages. Every type t randomizes among all messages according to the same probability distribution and the same $\beta_S \in \Delta(M)$. This is a Sequential Equilibrium, and it is called Babbling Equilibrium.

Let's see games in normal form. Suppose $T = \{t_H, t_L\}$, $M = \{t_H, t_L\}$ and $A_R = \{a, b\}$. Then we have the following matrix.

Rec|Send	t_H	t_L
a	a_1,a_1	0,0
b	0,0	b_2,b_2

In a separating equilibrium, the sender t_H sends a message t_H, and the receiver plays **a**. The sender t_L sends a message t_L, and the receiver plays **b**. Another equilibrium can be t_H sends a message t_L and the receiver plays **a**, and t_L sends a message t_H and the receiver plays **b**. The meaning of the messages is endogenous.

Suppose that $t_1 = \frac{1}{3}, t_2 = \frac{1}{3}$ and $t_3 = \frac{1}{3}$. Then we have the matrix.

Rec|Send	t_1	t_2	t_3
a	2,1	0,1	0,0
b	0,0	1,0	1,1

Types t_1 and t_2 send the message m with probability $\frac{1}{2}$ and the receiver plays **a**. Type 3 sends message m', and the receiver plays **b**. This is called a Semi-separating Equilibrium.

Let's see another case, where $t_1 = t_2 = \frac{1}{2}$.

Rec|Send	t_1	t_2
a	3,2	0,1
b	2,0	2,2
c	0,1	3,0

M is arbitrary. In Babbling Equilibrium, the receiver chooses **b**. Can I have an equilibrium such that after $\tilde{m}$, the receiver plays both **a** and **c** with positive probability? No because **a** and **c** will always be less than **b**. Indeed, $3\mu(t|\tilde{m}) = 3(1 - \mu(t|\tilde{m}))$ gives $\mu(t|\tilde{m}) = \frac{3}{2} < 2$ Therefore, the receiver, after any message can randomize between **ab**,**bc**, or play **a**,**b**,**c** but not **ac**.

Suppose there are two messages m and m' sent with positive probability, and the receiver reacts differently. Type t_1 cannot be indifferent between them, t_1 sends only m or m', therefore m' comes only from t_2.

Appendix : An introduction to Mechanism Design

So far, we have seen different examples of games and concepts of solutions. The main problem was that, given a game, of finding a suitable Nash Equilibrium or other concepts of solutions. Now the problem is different: that is, what we can achieve when agents have private information.

Example 19: an optimal auction

Imagine you are a seller and want to maximize your expected revenue. What is the auction mechanism you want to choose? This is an example of what is known as Mechanism Design.

Assume there is a seller who has an object to sell. The value is zero for her. There are also $n-$ potential buyers, each of them associated with a different type v_i. Types are private and independent. The payoff of type v_i is equal to $v_i - t_i$.

To find an optimal auction, fix a game and fix a BNE. The types are $(v_1, \ldots, v_n)$. The expected transfers of player i associated to these types are $t_i(v_1, \ldots, v_n)$. The probability i gets the good is $q_i(v_i, \ldots, v_n)$.

Notice that we just look at the expectation because the payoff is linear in the transfers. Furthermore, we need probabilities because the seller does not know what the buyers will do (perhaps they are mixing).

For the sake of simplicity, we can assume that the realization of v_i is v. Therefore, we can write:

$$T_i(v) = E_{v_{-i}}[t_i(v_1, \ldots, v_n)|v_i = v]$$

$$Q_i(v) = E_{v_{-i}}[q_i(v_1, \ldots, v_n)|v_i = v]$$

$$U_i(v) = v \cdot Q_i(v) - T_i(v)$$

The seller's revenues are written as:

$$R = E_{v_1,\dots,v_n}\Big(\sum_{t=1}^{n} t_i(v_1,\dots,v_n)\Big)$$
$$= \underbrace{\sum_{i=1}^{n}\int_0^{\infty} T_i(v) f_i(v) dv}_{\equiv E_{v_i}(T_i(v)) \equiv E_{v_i}\big(E_{v_i}(t_i(v_1,\dots,v_n)|v_i\big)} \tag{1}$$

If this is an equilibrium, a type cannot have a higher payoff mimicking another type. So for type v_i, it must be true:

$$\begin{aligned} U_i(v) =& v\cdot Q_i(v) - T_i(v) = \\ & v\cdot E_{v_{-i}}\Big(q_i(v_1,\dots,v_n)|v_i = v\Big) - E_{v_{-i}}\Big(t_i(v_1,\dots,v_n)|v_i = v\Big) \geq \\ & v\cdot E_{v_{-i}}E_{v_{-i}}\Big(q_i(v_1,\dots,v',\dots,v_n)|v_i = v\Big) - E_{v_{-i}}\Big(t_i(v_1,\dots,v',\dots,v_n)|v_i = v\Big) \end{aligned}$$

Another constraint is:

$$\begin{aligned} & E_{v_{-i}}\Big(q_i(v_1,\dots,v',\dots,v_n)|v_i = v\Big) = \\ & E_{v_{-i}}\Big(q_i(v_1,\dots,v',\dots,v_n)|v_i = v'\Big) = \\ & Q_i(v_i') \end{aligned}$$

For every v', v then we have:

- The Incentive Compatibility Constraint (IC) (see below):

$$v\cdot Q_i(v) - T_i(v) \geq v\cdot Q_i(v') - T_i(v') \tag{2}$$

- The individual rationality constraint (IR) (this assumption means that participation is voluntary):

$$U_i(v) \geq 0 \quad \forall v_i \in [0, V] \tag{3}$$

We can see that (2) + Individual Rationality for type 0 implies (3) $\forall v > 0$. To see this:

$$U_i(v) \equiv v\cdot Q_i(v) - T_i(v) \geq v\cdot Q_i(0) - T_i(0) \geq 0\cdot Q_i(0) - T_i(0) \equiv U_i(0) \geq 0$$

So we can write:

$$U_i(0) \geq 0 \tag{4}$$

Let's write the following proposition.

Proposition 15. *Let* $q_i : [0,V]^n \to [0,1]$ *such that* $\sum_{i_1}^n q_i(v_1,\ldots,v_n) \leq 1$ *for all* $(v_1,\ldots,v_n)$*, and write* $Q_i(v) = E_{v_{-i}}\Big(q_i(v_1,\ldots,v',\ldots,v_n)|v_i = v\Big)$.

Then exists a function

$$T_i : [0,V] \to \mathbb{R}$$

such that:

(i) $Q_i(\cdot), T_i(\cdot)$ *satisfies IC if and only if* $Q_i(\cdot)$ *is not decreasing* $\forall i$.

(ii) If $Q_i(\cdot), T_i(\cdot)$ *satisfy IC, then:*

$$U_i(v) = U_i(0) + \int_0^v Q_i(x)dx$$

and

$$T_i(v) = v \cdot Q_i(v) - \Big[U_i(0) + \int_0^v Q_i(x)dx\Big]$$

Proof. Let's start with (i) $\Leftarrow$. $\forall v, v'$ we can write:

$$U_i(v) = \underbrace{v \cdot Q_i(v) - T_i(v)}_{(1)} \geq \underbrace{v \cdot Q_i(v') - T_i(v')}_{(2)}$$

And

$$U_i(v') = \underbrace{v' \cdot Q_i(v') - T_i(v')}_{(3)} \geq \underbrace{v' \cdot Q(v) - T_i(v)}_{(4)}$$

We can see that:

$$(1) - (4) \geq (1) - (3) \geq (2) - (3)$$

Therefore:

$$(v - v')Q_i(v) \geq U_i(v) - U_i(v') \geq (v - v')Q_i(v')$$

Notice that if $v \geq v'$, then $Q_i(v) \geq Q_i(v')$. Therefore $Q_i(\cdot)$ is monotonic and weakly increasing. Therefore it is continuous almost everywhere. Take v as a continuity point at $Q_i(\cdot)$. Let $v \to v'$, and divide by $(v - v')$. Then we have $\frac{U_i(v) - U_i(v')}{(v - v')} \equiv U_i'(v)$ and:

$$Q_i(v) \geq U_i'(v) \geq Q_i(v')$$

By the Squeeze Theorem, taking the $\lim_{v \to v}$ of $Q_i(v)$ and $\lim_{v \to v'}$ of $Q_i(v')$, we have $Q_i(v) = U_i'(v)$.

Also:

$$\underbrace{(v - v')Q_i(v)}_{\in\,(0,1)} \geq U_i(v) - U_i(v') \geq \underbrace{(v - v')Q_i(v')}_{\in\,(0,1)}$$

This is a Lipschitz Function.[4] Then we can write:

$$|U_i(v) - U_i(v')| \leq |v - v'| \cdot 1$$

[4] A Lipschitz function is a function such that if it exists $k \geq 0$, $\forall x, y$ then $|f(x) - f(y)| \geq k \cdot |x - y|$

A Lipschitz Function is absolutely continuous. So we can write:

$$U_i(v) = U_i(0) + \int_0^v U'(x)dx =$$
$$U_i(0) + \int_0^v Q_i(x)dx$$

Let's see now (ii)$\Leftarrow$.

$$T_i(v) = v \cdot Q_i(v) - U_i(v) =$$
$$v \cdot Q_i(v) - \left[U_i(0) + \int_0^v Q_i(x)dx\right]$$

Let's see (i) $\Rightarrow$. Assume $Q_i(\cdot)$ weakly increasing. Define $T_i(v) = v \cdot Q_i(v) - \int_0^v Q_i(x)dx$ and $U_i(v) = \int_0^v Q_i(x)dx$.

Take $v > v'$.

$$U_i(v) - U_i(v') = \int_0^v Q_i(x)dx - \int_0^{v'} Q_i(x)dx$$
$$\int_0^v Q_i(x)dx \geq$$
$$\int_0^v Q_i(v')dx$$

Because $Q_i(\cdot)$ is weakly increasing. So:

$$U_i(v) - U_i(v') \geq$$
$$v \cdot Q_i(v) - v' \cdot Q_i(v)$$
$$U_i(v) \geq v \cdot Q_i(v') - \underbrace{\left[v' \cdot Q_i(v') - U_i(v')\right]}_{\equiv T_i(v)}$$

Take $v < v'$.

$$U_i(v) - U_i(v') =$$
$$\int_0^v Q_i(x)dx - \int_0^{v'} Q_i(x)dx =$$
$$-\int_0^v Q_i(x)dx \geq \int_0^v Q_i(v')dx =$$
$$Q_i(v')(v - v')$$

$\square$

To sum up, until now, we have fixed a game and a BNE. Further, we have written $Q_i(v)$ and $T_i(v)$ and shown that $U_i(v) = U_i(0) + \int_0^v Q_i(x)dx$. Assume two games and that the lower type has the same payoff ($U_i(0)$) and the same probability of taking the good ($\int_0^v Q(x)dx$). Then, the expected payoff is the same in the two games.

An easier way to build an equilibrium strategy is the following. Assume that types are i.i.d., that BNE are symmetric, and take the bidding strategy $b : [0, v] \to \mathbb{R}$. Then we can write $U_i(0) = 0$, $Q_i(v) = F(v)^{n-1}$ and:

$$U_i(v) = \int_0^v F(x)^{n-1} dx$$

Another way of writing an expected payoff is:

$$U_i(v) = [v - b(v)]F(v)^{n-1}$$

Therefore, we have:

$$\begin{aligned}
&[v - b(v)]F(v)^{n-1} = \int_0^v F(x)^{n-1} dx = \\
&vF(v)^{n-1} - b(v)F(v)^{n-1} = \int_0^v F(x)^{n-1} dx = \\
&- b(v)F(v)^{n-1} = \int_0^v F(x)^{n-1} dx - vF(v)^{n-1} = \\
&- b(v) = \Big(\int_0^v F(x)^{n-1} dx - vF(v)^{n-1} \Big) \cdot \frac{1}{F(v)^{n-1}} = \\
&b(v) = v - \int_0^v \frac{F(x)^{n-1}}{F(v)^{n-1}} dx = \\
&b(v) = v - \frac{1}{F(v)^{n-1}} \int_0^v F(x)^{n-1} dx
\end{aligned}$$

This is the optimal strategy for the bidder. Let's see for the seller. We have seen that the seller's revenue can be written as:

$$R = \sum_{i=1}^n \int_0^\infty T_i(v) f_i(v) dv$$

But we have shown that:

$$T_i(v) = v \cdot Q_i(v) - \Big[U_i(0) + \int_0^v Q_i(x) dx \Big]$$

Therefore, substituting in above and rearranging:

$$\begin{aligned}
R &= \sum_{i=1}^n \Big[\int_0^v v \cdot Q(v) f_i(v) dv - \int_0^v \Big(\int_0^v Q_i(x) dx) f(v) dv - \underbrace{\int_0^v U_i(0) f_i(v) dv}_{\equiv U_i(0)} \Big) \Big] \\
&= \int_0^v \Big(\int_0^v Q_i(x) dx \Big) f_i(v) dv
\end{aligned}$$

Using integration by parts, where:

$$g(v) = \int_0^v Q_i(x)dx$$

We have:

$$\int_0^v Q_i(x)dx \cdot F_i(v)\Big|_0^v - \int_0^v Q_i(v) \cdot F_i(v)dv =$$
$$\int_0^v Q_i(x)dx - \int_0^v Q_i(v)F_i(v)dv =$$
$$\int_0^v Q_i(v)(1 - F_i(v))fv$$

So:

$$R = \sum_{i=1}^{n} \Big[\int_0^v \Big(v - \frac{1 - F_i(v)}{f_i(v)} \Big) Q_i(v) f(v) dv - U_i(0) \Big]$$

Notice that the revenue of the seller depends only on $U_i(0)$ and $F(v)$. This leads us to an important result of auction theory, namely the Revenue Equivalence Principle. This simply stated that given two games, if the payoffs of the lowest type are the same, as well as their probability distributions, then the expected revenues for the seller are the same. This means that the expected revenues of a First Price Auction and Second Price Auction are the same.

Let's see now the optimal auction. Let's note that:

$$\sum_{i=1}^{n} E_v \Big(v - \frac{1 - F_i(v)}{f_i(v)} \Big) \underbrace{E_{v_{-i}} \Big(q(v_1, \ldots, v', 1 \ldots, v_n) | v_i = v \Big)}_{\equiv Q_i(v)}$$

By the Law of Iterated Expectations,[5] becomes:

$$E_{(v \ldots, v_i)} \Big(v_i - \frac{1 - F_i(v)}{f_i(v)} \Big) Q_i(v_1, \ldots, v_n)$$

So we can write:

$$R = E_{(v_1, \ldots, v_n)} \Big[\sum_{i=1}^{n} \Big(v_i - \frac{1 - F_i(v_i)}{f_i(v)} \Big) q_i(v_1, \ldots, v_n) \Big] - \sum_{i=1}^{n} U_i(0)$$

The seller wants to choose $q_1(\cdot), \ldots, q_n(\cdot)$, for all $v_1, \ldots, v_n$, with $\sum q_i(v_1, \ldots, v_n) \leq 1$ that maximizes R. Assuming $U_i(0) = 0$ for all i, then fix $v_1, \ldots, v_n$, the seller must choose $q_i(v_1, \ldots, v_n), \ldots, q_n(v_1, \ldots, v_n)$ in order to:

$$\max_{q_i(\cdot)} \sum_{i=1} \Big(\underbrace{v_i - \frac{1 - F_i(v_i)}{f_i(v)}}_{\equiv H_i(v_i)} \Big) q_i(v_1, \ldots, v_n)$$

[5] This states that:

$$E[E(x|y)] = E(x)$$

Then, we can demonstrate the following result.

Proposition 16. *Assume that $H(\cdot)$ is increasing for any player i. If there is an auction such that for every realization $(v_1, \ldots, v_n)$, the good is awarded to player i if and only if:*

$$H_i(v_i) \geq H_j(v_j) \quad \forall j \neq i$$

And $H_i(v) \geq 0$, and the good is not awarded to anyone if $H_i(v_i) < 0, \forall i$, then this auction is optimal (that is, it maximizes the seller's revenues).

Proof. I only demonstrate that $H(\cdot)$ is increasing. If $H_i(v) \leq 0$, then $Q_i(v_i) = 0$. If $Q_i(v_i) = Pr[H_i(v_i) \geq H_j(v_j)], \forall j \neq i$. Suppose that $v_i' > v_i$. Then $Q_i(v_i') = Pr[H_i(v_i') \geq H_j(v_j)]$, $\forall j \neq i$.If b is greater than a, then it is even more probable it is greater than c. Therefore $Q(v') \geq Q(v_i)$, and $Q(\cdot)$ is increasing because $H(\cdot)$ is increasing. □

Two final observations can be made. First, a trivial game is one where each player is required to reveal her type, and the seller constructs $H(\cdot)$ therefore. The good is awarded to the player with the highest $H(\cdot)$ so that an equilibrium strategy is that of telling the truth.

Second, in a symmetric environment, the good always goes to the player with the largest evaluation, but this is not true in general. Take, for example, two players, one with evaluation $v_1 = 1$ and a second whose evaluation is $v_2 = 1.1$ with probability 0.5 and evaluation $v_2 = 3.000.000$ with probability 0.5. In this case, v_2 always takes the good, but the maximum the seller can charge is 1.1.

Mechanism Design: general principles

As seen, Mechanism Design concerns what we can achieve when agents have private information. In general, we have an environment, G, with the following variables:

- a set of alternatives;
- a set of types, $T_1, \ldots, T_n$;
- a probability distribution function;
- a set of payoffs functions

Then we can write:

$$G = (A, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$$

Notice that the set of alternatives is not the set of actions, because we are not in a game, we must find a game. An example of A can be $\{0, 1\} \times \mathbb{R}$ in the case of just a buy/sell situation, where 0 indicates that the good is not purchased and 1 otherwise. Therefore, we write the payoff function as $u(a, t)$, i.e., the payoff depends on the alternative and the type.

Thinking of the seller (or the planner or the policy official) as a principal, the principal is interested in implementing an outcome function:

$$f : T \longrightarrow \Delta(A)$$

An example can be a social planner that wants to give the good to the type with the largest evaluation. The problem is that the principal does not know the types of the agents.

The principal wants to develop a Mechanism (S, γ) where $S = S_1 \times S_2 \cdots \times S_n$ (S_i is the set of actions of the players) and:

$$\gamma : S \longrightarrow \Delta(A)$$

The Mechanism induces a game, that is, what the players can do and how the outcome is reached.

We can define a Bayesian Game as follows:

$$G^{(S,\gamma)} = \Big(T_1, \dots, T_n, p, S_1, \dots, S_n, u_1^{(s,\gamma)}, \dots, u_n^{(s,\gamma)}\Big)$$

And the payoff function as:

$$u_i^{(s,\gamma)}(s, t) = \sum_{a \in A} \gamma(a|s) u_i(a, t)$$

Where $S = S_1, \dots, S_n)$ and $t = t_1, \dots, t_n$.

We can define pure strategies as:

$$h_i : T_i \longrightarrow S_i$$

And mixed strategies as:

$$\sigma_i : T_i \longrightarrow \Delta(S_i)$$

Then we can write:

$$\begin{gathered} u_i^{(s,\gamma)}(\sigma_1(t_1), \dots, \sigma_n(t_n), t_1, \dots, t_n) = \\ \sum_{s \in S} \Big(\prod \sigma_j(s_j|t_j)\Big) \gamma(a|s) u_i(a, t) = \\ \sum_{s \in S} \Big(\prod \sigma_j(s_j|t_j)\Big) \underbrace{u_i^{(s,\gamma)}(s, t)}_{\gamma(a|s)u_i(a,t)} \end{gathered}$$

A particular class of games is called Direct Mechanism (or Direct-relation Mechanism). These are games where $S_i = T_i$, $\forall i$. That is, where the set of actions is equal to the set of types. A Mechanism of this type is the game "tell me your type." An auction clearly does not belong to this class. Indeed it is an example of an indirect mechanism.

Then, we have the following definitions.

Definition .0.6 (Implementability)**.** An outcome function f is implementable via a certain solution concept (a Bayesian Nash Equilibrium or Strategic Dominance) if there exists a mechanism (S, γ) such that $(f(t))_{t \in T)}$ is the optimal outcome of $G^{(s,\gamma)}$ using the corresponding solution concept

Definition .0.7. f is implementable in Bayesian Nash Equilibrium if there exists a mechanism (S, γ) and a Bayesian Nash Equilibrium σ^* of $G^{(S,\gamma)}$ such that $\forall a \in A$ and $\forall t = (t_1, \ldots, t_n) \in T$

$$\sum_{s \in S} \Big(\prod_{j=1}^{n} \sigma_j^*(s_j|t_j) \Big) \gamma(a|s) = f(a|t)$$

Notice that f is given (what we have to show). The left term refers to the probability that player j plays s given t_j

Defining F as the set of outcome functions, a natural question is to ask if it is implementable. This question has a simple answer due to an important result called the Revelation Principle (due, among the others, to Roger Myerson).

Let's define another important notion, that of Incentive Compatibility.

Definition .0.8. (Incentive Compatibility) Given a direct mechanism (T, μ), $\mu : T \longrightarrow \Delta(A)$ is incentive compatible if true-telling is a Bayesian Nash Equilibrium, that is, $\forall i, \forall t, t' \in T$:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \mu(a|t_i, t_{-i}) u_i(a, t_i, t_{-i}) \geq \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \mu(a|t'_i, t_{-i}) u_i(a, t_i, t_{-i})$$

Therefore, we can state the Revelation Principle.

Proposition 17 (Revelation Principle)**.** *$f : T \longrightarrow \Delta(A)$ is implementable in Bayesian Nash Equilibrium if and only if (T, f) is incentive compatible.*

Proof. ($\Rightarrow$) If we implement f with the direct mechanism, we can do that for every mechanism. This is obvious. Let's look at the next part.

($\Leftarrow$) If we implement f with any mechanism, we can do it with the direct mechanism. Suppose f is implementable and σ^* is a Bayesian Nash Equilibrium of $G^{(s,\gamma)}$ such that $\forall a \in A, \forall t \in T$:

$$\sum_{s \in S} \Big(\prod \sigma_j^*(s_j|t_j) \Big) \gamma(a|s) = f(a|t)$$

Now we construct a Direct Mechanism (T, μ), $\forall a, \forall t$

$$\mu(a|t) = \sum_{s \in S} \Big(\prod \sigma_j^*(s_j|t_j \Big) \gamma(a|s)$$

Clearly $\mu(t) = f(t)$, $\forall t$, $\mu(t) \in \Delta(A)$. We must show the Incentive Compatibility of (T, μ), that is, $\forall i, \forall t_i, t'_i \in T_i$

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \mu(a|t_i, t_{-i}) u_i(a, t_i, t_{-i})$$

Since $\mu(a|t) = \sum_{s \in S} \left(\prod \sigma_j^*(s_j|t_j) \right) \gamma(a|s)$, we can write:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \left(\sum_{s \in S} \left(\prod \sigma_j^*(s_j|t_j) \right) \right) \gamma(a|s) u_i(a, t_i, t_{-i})$$

Since $\sum_{s \in S} \left(\prod \sigma_j^*(s_j|t_j) \right) \gamma(a|s) u_i(a, t_i, t_{-i}) \equiv u_i^{(s,\gamma)}(\sigma_1^*(t_1), \ldots, \sigma_i^*(t_i), \sigma, \sigma_n^*(t_n), t_i, t_{-i})$ we can write:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i^{(s,\gamma)}(\sigma_1^*(t_1), \ldots, \sigma_i^*(t_i), \sigma, \sigma_n^*(t_n), t_i, t_{-i})$$

Since in equilibrium, we must have $\sigma_i^*(t_i) \geq \sigma_i^*(t_i')$, this is greater and equal than:

$$\begin{aligned}
&\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_i^{(s,\gamma)}(\sigma_1^*(t_1), \ldots, \sigma_i^*(t_i'), \sigma, \sigma_n^*(t_n), t_i, t_{-i}) = \\
&\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \left(\sum_{s \in S} \sigma_i^*(t_i') \prod_{j \neq i} \sigma_j^*(s_j|t_j) \right) \gamma(a|s) u_i(a, t_i, t_{-i}) = \\
&\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} \mu(a|t_i', t_{-i}) u_i(a, t_i, t_{-i})
\end{aligned}$$

This concludes the proof. □

Therefore, we can redefine Implementability in Bayesian Nash Equilibrium as follows.

Definition .0.9. (Implementability in Bayesian Nash Equilibrium) Given a situation $G = (A, T_1, \ldots, T_n, p, u_1, \ldots, u_n)$, $f : T \longrightarrow \Delta(A)$ is implementable in Bayesian Nash Equilibrium if and only if (T, f) is Incentive Compatible, that is, $\forall i, \forall t_i, t_i'$:

$$\sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} f(a|t_i, t_{-i}) u_i(a, t_i, t_{-i}) \geq \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) \sum_{a \in A} f(a|t_i', t_{-i}) u_i(a, t_i, t_{-i})$$

What the revelation principle states is that to implement some outcome, one must check only for incentive compatibility, that is, restrict the search only to those mechanisms where they are willing to reveal their private information.

However, these equilibria are not necessarily unique. If true-telling is weakly dominant, but it is not a Bayesian Nash Equilibrium, then it is said to be Strategy-proof.

Finally, notice that in some situations, it can be assumed that participation is voluntary. Therefore, a further condition is needed, that is, individual rationality: namely, given a reservation payoff for all i, $w_i(t)$ $\forall t_i$, we have:

$$\sum_{t_{-i} \in T_{-i}} p(t_i|t_{-i}) \sum_{a \in A} f(a|t_i, t_{-i}) u_i(a, t_i, t_{-i}) \geq w_i(t)$$