Microeconomics
Notes on Production Theory

gianluca.damiani@carloalberto.org

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1 Production Sets

Production theory refers to the supply side of the economy. This side is made up of a series of productive units, called "firms", of which only what can they do is of interest. Then the firm is seen merely as a "black box", able to transform inputs in outputs.

Let's consider an economy with L commodities. A *Production Vector* is a vector $y = (y_1, \dots y_L) \in R^L$ that describes the net outputs of the L commodities from the production process. A convention is that of writing the positive numbers in the vector as outputs, and the negative as inputs.

The set of feasible production vectors from which feasible plans can be arranged is known as the *Production Set*, $Y \subset R^L$. Any $y \in Y$ is possible. This set is primarily limited by technological constraints.

This set can also be described by using a function, called *Transformation Function*:

$$Y = \{y \in R^L : F(y) \geq 0\}$$

Furthermore, F(.) $= 0$ if and only if y is a boundary element of Y, that is an element of the *Transformation Frontier* of Y. These are represented in Figure 1.

If $F(.)$ is differentiable, and if $\bar{y}$ belongs to the transformation frontier (i.e. $F(\bar{y}) = 0$, then, for any commodities l and k, we can write the *Marginal Rate of Transformation of good k for good l at $\bar{y}$*:

$$MRT_{l,k}(\bar{y}) = \frac{\frac{\partial F(\bar{y})}{\partial y_l}}{\frac{\partial F(\bar{y}}{\partial y_k}}$$

This represents how much the net output of good k can increase if the firm decreases the net output of good l by one marginal unit. The slope of the transformation frontier at $\bar{y}$ in Figure 1 is $-MRT_{1,2}(\bar{y})$.

We can describe any production model in terms of output-technology, that is by using the idea of *Production Function* $f(z)$ (in the case of a single output). This gives the maximum amount of output that can be produced by using inputs $(z_1, \dots, z_L - 1) \geq 0$.

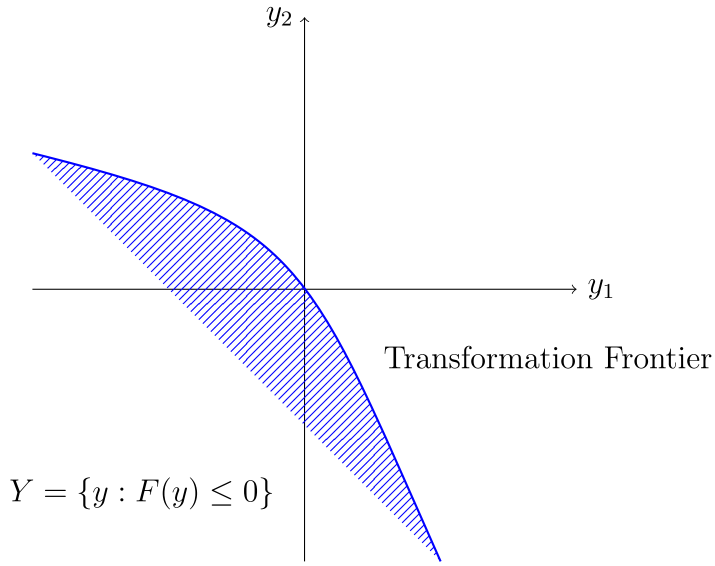


Figure 1: The Production Set and the Transformation Frontier

Holding fixed the level of output, we can define the *Marginal Rate of Technical Substitution of input* l *for input* k *at* $\bar{z}$, as follows:

$$MRTS_{l,k}(\bar{z}) = \frac{\frac{\partial f(\bar{z})}{\partial z_l}}{\frac{\partial f(\bar{z}}{\partial z_k}}$$

This represents the additional amount of input k that must be used to keep the output at the level $\bar{q} = f(\bar{z})$, when the amount of l is decreased marginally. This is for production theory the analogous of the Marginal Rate of Substitution in Consumer Theory, whereas in the latter the utility level was aimed to be kept constant. The $MRTS$ is a renaming of the MRT in the case of a single output-many input technology.

The Production Sets have a series of commonly assumed properties.

1. Y is *non-empty*

2. Y is *closed*: i.e, Y contains its boundary. The limit of a sequence made up of $y_n \in Y$ is still in Y.

3. *No Free-Lunch*: if $y \in Y$ and $y \geq 0$. This property says that y cannot produce any output either.

4. *Possibility of Inaction*: $0 \in Y$. In words, complete shutdown is possible.

5. *Free Disposal*: if $y \in Y$ and $y' \geq y$, then $y' \in Y$. In words, an extra-amount of inputs, or outputs, can be eliminated without any cost.

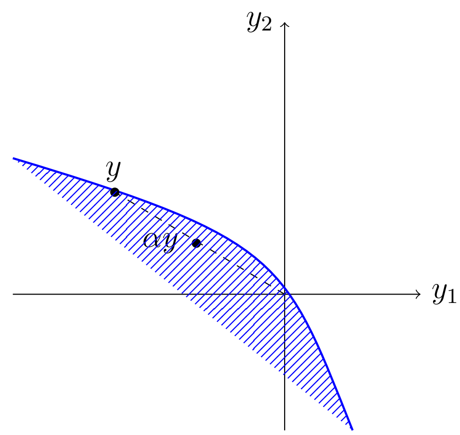


Figure 2: Non-Increasing Returns to Scale

6. *Irreversibility*: if $y \in Y$ and $y \neq 0$. Then $-y \neq Y$. It is impossible to transform an amount of output into the same amount of input that has been used to generate it.

7. *Non-increasing Returns to Scale*: if for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha \in [0, 1]$. See Figure 2

8. *Nondecreasing Returns to Scale*: if for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha > 1$. See Figure 3

9. *Constant Return to Scale*: if for any $y \in Y$, we have $\alpha y \in Y$ for all $\alpha > 0$. This means that Y is a Cone. See Figure 4

10. *Additivity (or Free-Entry)*: suppose that $y \in Y$ and $y' \in Y$. Then also $y' + y \in Y$. That is, for instance, that $ky \in Y$ for any $k \in N$.

11. *Convexity*: this means that a production set is a convex set. If $y, y' \in Y$ and $\alpha \in [0, 1]$ then $\alpha y + (1 - \alpha)y' \in Y$.

12. *Y is a convex cone*: this derives from the conjuction between convexity and constant returns to scale (see Figure 4 again). Y is a Convex Cone if, for any production vector y, $y' \in Y$ and constants $\alpha \geq 0$ and $\beta \geq 0$, we have $\alpha y + \beta y' \in Y$.

From these properties two results derive. First, differently from consumer theory, when from convex preferences derive a quasi-concave utility function (as well as a convex demand set), in the case of production, for a single-output technology, if the production function $f(z)$ is concave, then the production set Y is convex.

The second, is a proposition:

Proposition 1 (MWG 5B1)**.** *The production set Y is additive and satisfies the non-increasing returns condition if and only if it is a convex cone.*

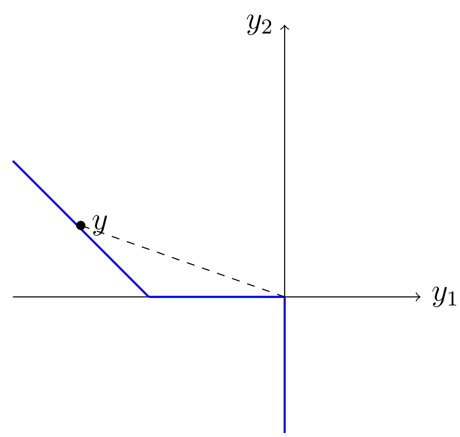


Figure 3: Non-Decreasing Returns to Scale

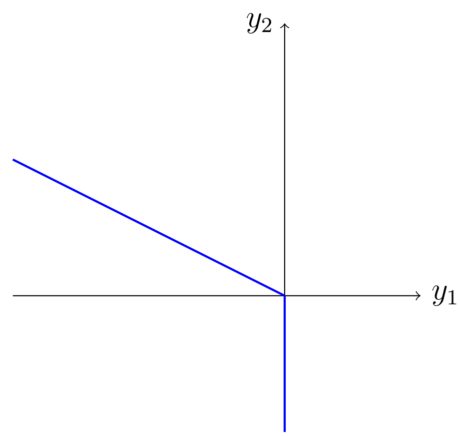


Figure 4: Constant Returns to Scale

Proof. The definition of a convex cone directly implies the non-increasing returns and additivity properties. Let's show that if nonincreasing returns and additivity hold, then for any $y, y' \in Y$ and $\alpha > 0$, and $\beta > 0$m we have $\alpha y + \beta y' \in Y$. Let k be any integer such that $k > \max \alpha, \beta$. By additivity, $ky \in Y$ and $k'y \in Y$. Since $\frac{\alpha}{\beta} < 1$ and $\alpha y = (\frac{\alpha}{k} \cdot ky$, by the definition of non-increasing returns, $\alpha y \in Y$. Similar for $\beta y'$. And, by additivity $\alpha y + \beta y' \in Y$ □

2 The Profit Maximization Problem

As in the study of the consumer demand, we assume that there is a vector of prices for the L goods, $p = (p_1, \ldots p_L) >> 0$, and that these prices are independent of the production plans of each firm (*price-taking assumption.*

Letting aside all other issues concerning the institutional role of the firm, throughout this theory we assume that firms are only profit maximizers, and that Y satisfies *non-emptiness, closedeness* and *free-disposal.*

We can write the Profit Maximization Problem (PMP) as follows. Given a price vector $p >> 0$ and a production vector $y \in R^L$, we can write the firm's Problem as follows:

$$\max_{y} p \cdot y \quad \text{s.t.} \;\; y \in Y$$

In the simplest case, of only one output, we can write:

$$\max_{z \geq 0} p \cdot f(z) - w \cdot z \tag{1}$$

The z^* maximizes profits given (p, w) if it solves the problem above. The F.O.C. are:

$$p \frac{\partial f(z^*)}{\partial z_l} \geq w_l$$

With equality if $z_l^* > 0$. In words, the marginal product of every input l actually used must equal its prices in terms of output $\frac{w_l}{p}$.

In the case of two inputs l, k, and $(z_l^*, z_k^*) >> 0$, the F.O.C implies that the $MRTS_{l,k} = \frac{w_l}{w_k}$. That is:

$$\frac{\frac{\partial f(z^*)}{\partial x_l}}{\frac{\partial f(x^*)}{\partial z_k}} = \frac{w_l}{w_k}$$

The marginal rate of technical substitution between l, k is equal to their price ratio (the economic rate of substitution between them). If the price set Y is convex, the F.O.C are sufficient for the determination of a solution of the Profit Maximization Problem.

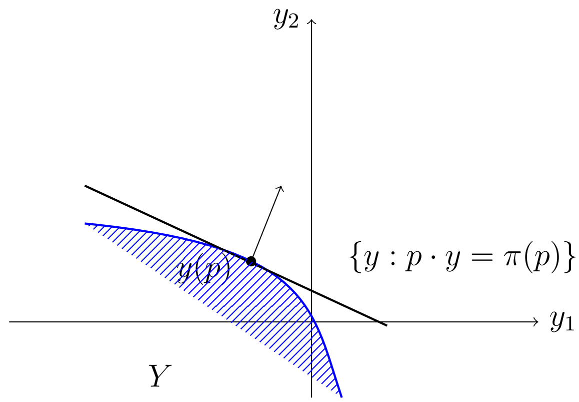


Figure 5: The profit maximization problem

Given a production set Y, the firm's *profit function* $\pi(p)$ associates to every p the amount which maximizes the PMP. Similarly, we define the firm's *supply correspondence* at p, $y(p)$ as the set of profit-maximizing vector, i.e. $y(p) = \{y \in Y : p \cdot y = \pi(p)\}$. See Figure 5.

An interesting result says that, in general, if the production set Y exhibits non-decreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$. In other words, if returns are constant or increasing, then profits cannot be positive. To see this, assume they are, that is: $0 < \pi < \infty$. So $y^* \in y(p)$ and $\pi(p) = p \cdot y^*$. Furthermore, $y^* \in Y$. Take now $y^{*}* = 2y^*$. By non-decreasing returns $2y^* \in Y$. And therefore $p \cdot y^* = p \cdot 2y^* = 2 \cdot p \cdot y^* = 2\pi(p)$. Which is clearly a contradiction with $\pi(p)$ being the maximized profit.

An important proposition list the properties of the profit function and supply correspondence.

Proposition 2 (MWG 5.C.1)**.** *Suppose that $\pi(.)$ is the profit function of the production set Y and $y(.)$ is the associated supply correnspondence. Assume also that Y is closed and satisfied the free disposal property. Then:*

1. *$\pi(.)$ is homogeneous of degree one*

2. *$\pi(.)$ is convex*

3. *If Y is convex, then $Y = \{y \in R^L : p \cdot y \leq \pi(p), \forall \;\; p >> 0\}$*

4. *$y(.)$ is homogeneous of degree zero*

5. *If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued (if nonempty)*

6. *if $y(p)$ is a function, and $\pi(p)$ is differentiable, then:*

$$\frac{\partial \pi(p)}{\partial p_i} = y_i(p)$$

This is called Hotelling's Lemma.

7. *If $y(p)$ is differentiable, then $D_y(p) = D^2\pi(p)$ is a symmetric and positive semi-definite matrix with $D_y(p)p = 0$.*

Proof. Let's prove only convexity of $\pi(p)$. First, note that, given p and $y(p)$, $\pi(p)$ can be written as $p \cdot y(p)$. The same for p', $y(p')$, then $\pi(p') = p'y(p')$. Take a $p'' = \alpha p + (1-\alpha)p'$. We want to prove that: $\pi(p'') \leq \alpha\pi(p) + (1-\alpha)\pi(p)$. Furthermore, we know that for all $y \in Y$, $py \leq py(p)$, and the same for p'. Choose a p'' s.t. $p''y = [\alpha p + (1-\alpha)p']y = \alpha py + (1-\alpha)p'y$. Then $p''y \leq \alpha py(p) + (1-\alpha)p'y(p)$. And $p''y \leq \alpha\pi(p) + (1-\alpha)\pi(p')$. Finally: $\max_{y \in Y} p''y \leq \alpha\pi(p) + (1-\alpha)\pi(p') = \pi(p^*)$. □

Property 3. simply says that $\pi(p)$ is, for firm's theory, the equivalent that the indirect utility function is for the consumer theory. And thanks to the Hotelling's Lemma, it can be easily used to recover firm's supply.

The positive semi-definiteness of the matrix $D_y(p)$ is the general mathematical expression of *Law of Supply: Quantities moves in the same directions of prices. If the price of an output increases, then the quantity supplied does the same. If the price of an input increases, the demand for the input decreases.* An important remark: since there is no budget constraint, in contrast with the demand theory, there are no compensation requirements. In other words, no wealth effects, just substitution effects.

3 The Cost Minimization Problem

Cost Minimization is a necessary condition for profit maximization. Indeed a firm choosing a profit-maximizing production plan implies that there is no way to produce the same amounts of outputs at a lower total input cost.

Let's focus on the single-output case. z is a nonnegative vector of inputs, $f(z)$ is the production function, q the amounts of output, and $w >> 0$ the vector of input prices. The *Cost Minimization Problem* (CMP) is the following (assuming free disposal of output):

$$\min_{z \geq 0} w \cdot z \quad \text{s.t. } f(z) \geq q \tag{2}$$

The optimized value of the CMP is given by the Cost Function $c(w, q)$. The optimizing set of input choices, the *Conditional Factor Demand Correnspondence* (or function), is $z(w, q)$.

If z^* is optimal in the CMP, and if the production function $f(.)$ is differentiable, then for some $\lambda \geq 0$ the following first order conditions hold for every input $l = 1, \ldots, L-1$:

$$w_l \geq \lambda \cdot \frac{\partial z^*}{\partial z_l}$$

with equality in $z_l^* > 0$. As for the PMP the condition above implies that for any two inputs l, k with $(z_l, z_k) >> 0$, the $MRTS_{lk} = \frac{w_l}{w_k}$.

This exhibits a strong analogy with the Expenditure Minimization Problem in Consumer Theory.

The following proposition collects the main properties of the cost function and Conditional Factor Demand correspondence.

Proposition 3 (MWG 5.C.2)**.** *Suppose that $c(w, q)$ is the cost function of a single-output technology Y with production function $f(.)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then:*

1. *$c(.)$ is homogenous of degree one in w and non-decreasing in q*

2. *$c(.)$ is a concave function of w*

3. *If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q) \forall w >> 0\}$*

4. *$z(.)$ is homogenous of degree zero in w.*

5. *If the set $\{z \geq 0 : f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. If $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is single-valued.*

6. *if $z(w, q)$ consists of a single point, then $c(.)$ is differentiable with respect to w at $\bar{w}$ and:*
$$\frac{\partial c(\bar{w}, q)}{\partial w} = z(\bar{w}, q)$$
   This is called Shephard's Lemma.

7. *If $z(.)$ is differentiable at $\bar{w}$, then $D_w z(\bar{w}, q) = D_w^2 c(\bar{w}, q)$ is a symmetric and negative semidefinite matrix with $D_w z(\bar{w}, q)\bar{w} = 0$*

8. *If $f(.)$ is homogeneous of degree one (i.e. it exhibits constant returns to scale), then $c(.)$ and $z(.)$ are homogenous of degree one in q.*

9. *If $f(.)$ is concave, then $c(.)$ is a convex function of q (in particular, marginal costs are non-decreasing in q)*

Using the cost-function, the firm's PM problem can be restate as follows:

$$\max_{q \geq 0} p \cdot q - c(w, q).$$

The F.O.C for q^* to be profit maximizing is then:

$$p - \frac{\partial c(w, q^*)}{\partial q} \geq 0$$

with equality if $q^* > 0$. In words, at an interior optimum, *price equals marginal cost.*

4 The Geometry of Cost and Supply in the Single-Output Case

We see the relationship between a firm's technology, its cost function and its supply behavior in the special case of a single output.

Let's denote the amount of output by q and the vector of factor prices constant at $\bar{w}$. The firm's cost function as $C(q)$. The *Average Cost Function* as $\frac{C(q)}{q}$. And finally, assuming the function is differentiable, the *marginal cost function* as $C'(q) = \frac{dC(q)}{dq}$.

Recall that profit-maximizing output levels $q \in q(p)$ satisfy $p \leq C'(q)$ (with equality if $q > 0$). If Y is convex, then $C(.)$ is convex too (from Proposition 3]. Therefore, marginal costs are non-decreasing. In this case, the satisfaction of the condition above is also sufficient for establishing that q at p is a profit-maximizing output.

Let's see three different cases.

4.1 Strictly Decreasing Returns to Scale

The figures above depicts the production set, the cost function and average and marginal cost functions for a case of decreasing returns to scale. Note that this case, as well as the following, exhibits convex production sets Y. Therefore the supply locus in each case coincides with the (q, p) combinations that satisfy the first order condition: prices equal marginal costs.

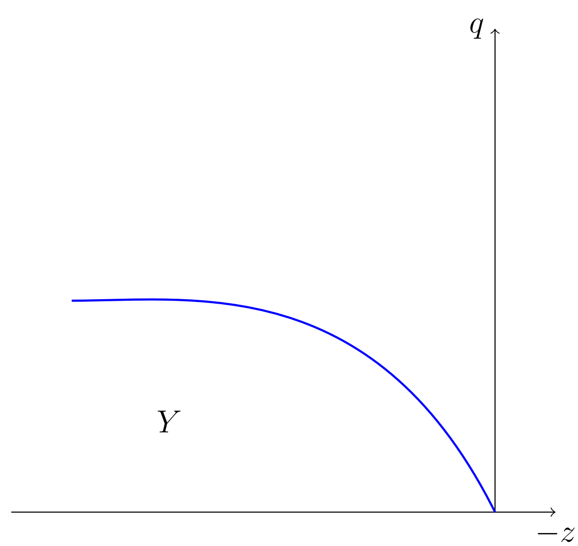


Figure 6: Production Set

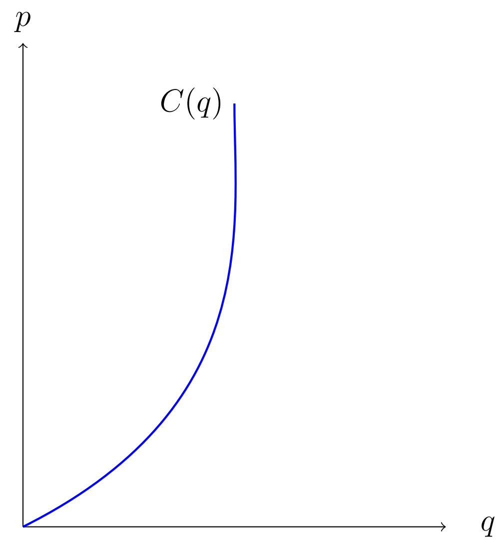


Figure 7: Cost Function

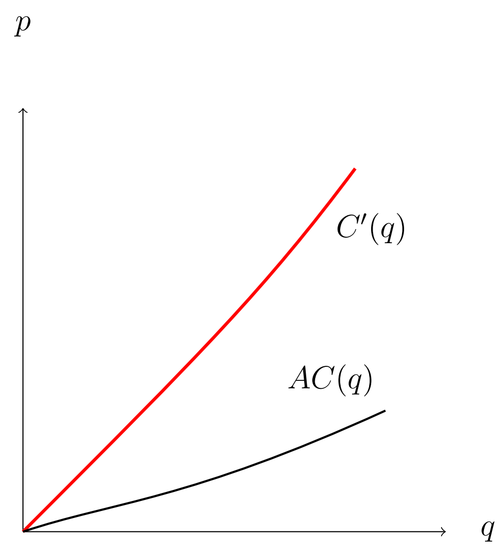


Figure 8: Average Cost, Marginal Cost and Supply

4.2 Constant Returns to Scale

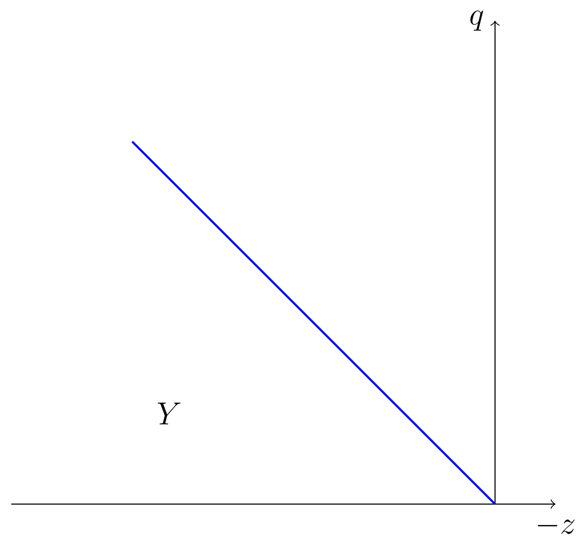


Figure 9: Production Set

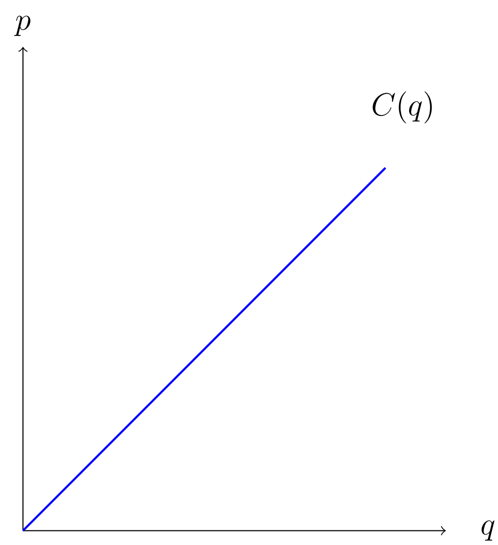
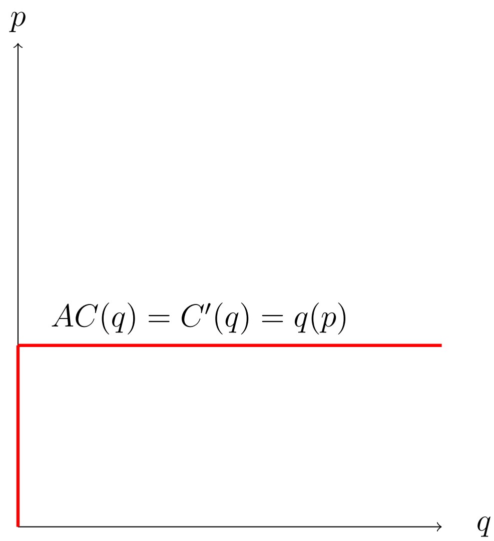


Figure 10: Cost Function



4.3 Fixed Costs and Decreasing Returns to Scale

Fixed costs are an important source of non-convexities. The figures below show non-sunk fixed costs (i.e. the inaction is possible). These parallels the cases above. Total cost is now of the form: $C(0) = 0$ and $C(q) = C_v(q) + K$ for $q > 0$. $K > 0$ and $C_v(q)$,is convex. The firm will produce a positive amount of output only if its profits is sufficient to cover not only its variable costs, but also the fixed cost K.

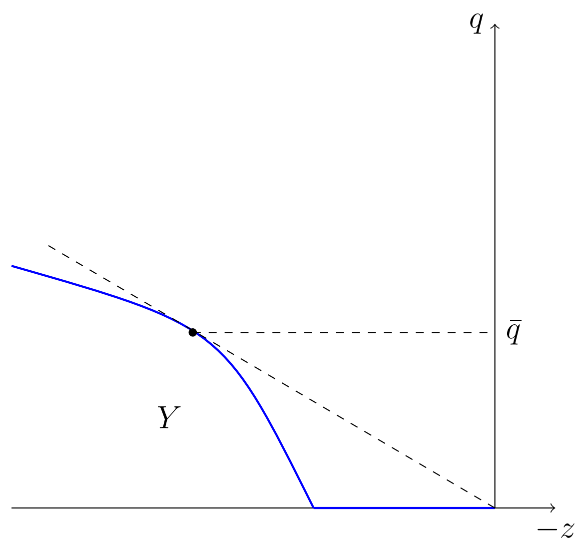


Figure 11: Production Set

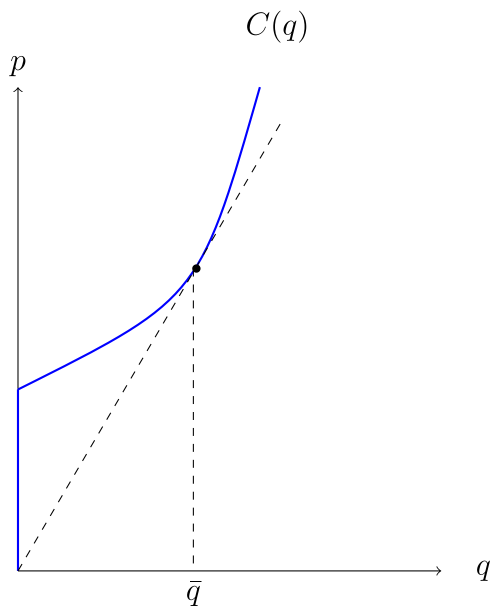


Figure 12: Cost Function

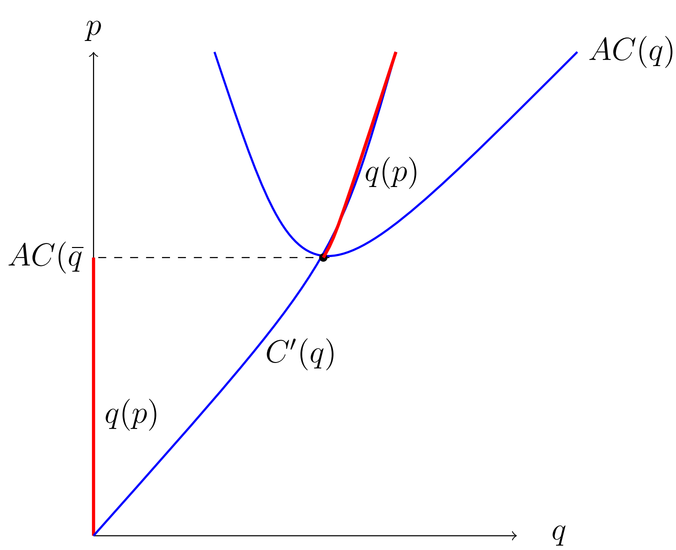


Figure 13: Average Cost, marginal cost, and supply

4.4 Fixed Costs and Constant Returns to Scale

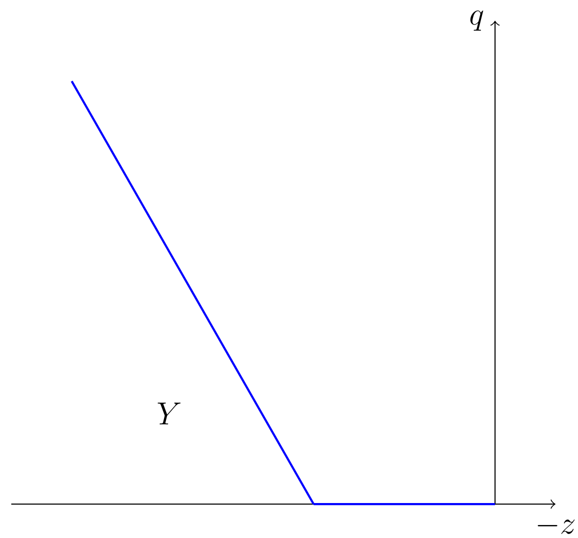


Figure 14: Production Set

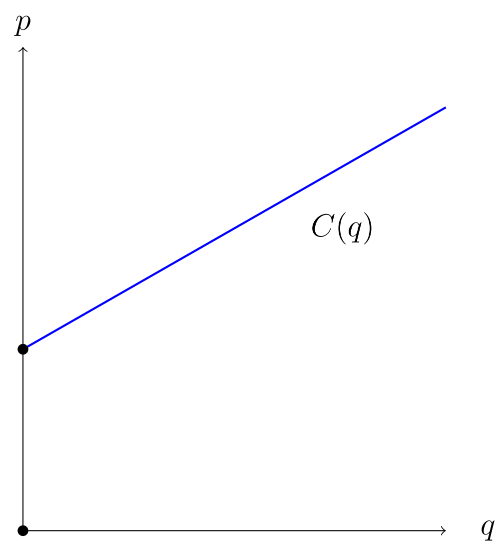


Figure 15: Cost Function

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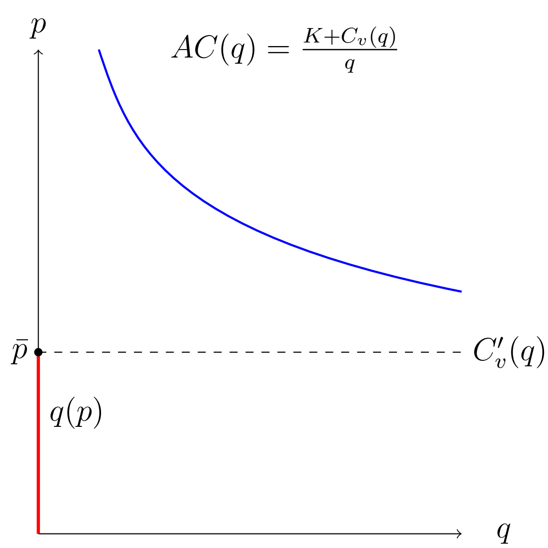

Figure 16: Average Cost, marginal cost, and supply