Notes on Microeconomics Demand Theory and General Equilibrium

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Chapter 1

Preferences and Choices

1.1 Introduction

Economics models human activity as the interaction of individual agents pursuing their private interests. Therefore, the building block of economic theory is the study of individual decision-making.

More in detail, the real starting point is the theory of decision-making in the most abstract possible setting. This is based on the idea of individual preferences, or choices, over a set of alternatives. For each alternative, the rational decision-maker can generate a ranking and, therefore, pick what are her preferred alternatives.

In general, the primitives of the decision-making problem are:

- The set of alternative X, where each $x \in X$ is supposed to be a complete description of the decision maker's preferences.
- A preference relation $\succeq$ on X. This means, $\forall\, x, y \in X$, if $x \succeq y$, then x is **weakly preferred** to y. $\succeq$ is assumed to be reflexive, so:

$$x \succeq x$$

- The **feasible set**. This, defined as $B \subseteq X$, is the set of alternatives that are feasible to the decision-maker

Once defined a preference relation and a feasible set, we can define a further object, the set of most preferred chosen alternatives called the **maximal set**:

$$C^*(B, \succeq) \equiv \Big\{x \in B : x \succeq y, \forall y \in B\Big\}$$

Notice that:

1. $C^*(B, \succeq)$ can be empty or contain multiple alternatives
2. The actual choice is one element of $C^*(B, \succeq)$.

3. Notice that the decision maker's problem does not depend on the feasible set.

There are two ways to model individual choice behavior. One is based on preferences, i.e., the decision-makers' tastes. On these are imposed rationality axioms, and then we analyze the consequences of these preferences for her choices. The second approach is based on the agent's actual choices. What matters, in this second case, is if choices are consistent. Notice, however, that these two approaches are not entirely separated (even if they belong to two different attempts to tackle the problem of modeling individual agency, one based on utility functions, the other on the observation of actual choices). Instead, several successful results have been provided that link the optimal decision in terms of preferences (and, therefore, the maximization of a utility function) and consistency.

Let's start with the so-called "preference-based" decision-making.

1.2 Rational Choice

1.2.1 Preference-based decision making

In this approach, we define **rationality** by two axioms of the preference relation $\succeq$.

Definition 1.2.1. If $\succeq$ satisfies the following axioms:

1. **completeness**: for all $x, y \in X$, $x \succeq y$ or $y \succeq x$
2. **transitivity**: for all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$

Then, the preference relation is said to be **rational**

Two further relations, from $\succeq$, can be defined.

Definition 1.2.2. Given a $\succeq$ on X, then:

1. **strict preference**: $x \succ y$ if and only if $x \succeq y$ and $y \not\succeq x$.
2. **indifference** $x \sim y$ If and only if $x \succeq y$ and $y \succeq x$

Lemma 1.2.1. *If $\succeq$ is rational then:*

1. *both $\succ$ and $\sim$ are transitive*
2. *for any $x, y, z \in X$, then:*
$$x \succ y \succeq z \Rightarrow x \succ z$$
and
$$x \succeq y \succ z \Rightarrow x \succ z$$
3. *$\succ$ is irreflexive ($x \succ x$ never holds). $\sim$, on the contrary, is reflexive.*

Proof. Let's see 1). We want to show $x \succ y \succ z$. Take $x \succ y$. This means $x \succeq y$ and $y \not\succeq x$. Take $y \succ z$. Similarly, this means $y \succeq z$ and $z \not\succeq y$. Then, by transitivity of $\succeq$, $x \succeq y \succeq z$. We need to show the not $z \succeq x$ part. By contradiction, assume $z \succeq x$. By transitivity of $\succeq$ we have $z \succeq y$. But we have shown $y \succ z$, so we have reached a contradiction.

Let's see 2).

Let's see 3). $x \succ x$ means $x \succeq x$ and $x \not\succeq x$. A contradiction. $x \sim x$ means $x \succeq y$ and $y \succeq x$, which can be, by completeness of $\succeq$. □

Theorem 1.2.2. *Suppose $\succeq$ is rational. Then, for every finite non-empty set B,*

$$C^*(B, \succeq) \neq \emptyset$$

Proof. Let's use mathematical induction (since B is assumed to be finite). Suppose $B = \{x\}$. Then, since $x \succeq x$, by completeness, $C^*(B, \succeq)$ is not empty. Take $B = \{x_1, x_2, \ldots, x_i, \ldots\}$ $i = 1, \ldots, n$. Take $B \cup \{x_{n+1}\}$. Since $\succeq$ is complete, then we have one of the two following cases:

1. $x_n \succeq x_{n+1}$, so $x_n \in C^*(B, \succeq)$
2. $x_{n+1} \succeq x_n$, so $x_{n+1} \in C^*(B, \succeq)$

In any case $C^*(B, \succeq)$ is not empty, □

Notice, however, that if B is infinite, $C^*(B, \succeq)$ can be empty. Furthermore, transitivity is a sufficient condition to guarantee nonemptiness of $C^*(B, \succeq)$, but is not necessary. Indeed, a weaker condition is enough, **acyclicity**.

Definition 1.2.3. A preference relation $\succeq$ is acyclic if, for all $\{x, y, z, \ldots, u, v\}$,

$$x \succ y \succ z \ldots u \succ v \Rightarrow x \succeq v$$

Notice that transitivity implies acyclicity, but the contrary is not true.

Theorem 1.2.3. *Suppose $\succeq$ are rational. Then, the following statements are equivalent:*

1. *$\succeq$ is acyclic*
2. *$C^*(B, \succeq) \neq \emptyset$, for all non-empty an finite B.*

Proof. □

Notice, however, that transitivity is a very strong assumption. It is easy to think about real-life situations where transitivity does not hold. A standard and intuitive example is that of just perceptible differences. Namely, if an individual is asked to choose between two cups of coffee, where the difference is just one grain of sugar, she can obviously be unable to taste the difference. However, by adding sugar, she will be able to distinguish a cup of sugarless coffee from a cup of coffee with sugar. Other serious attacks on the empirical relevance of transitivity have come from behavioral economists with their theory of framing. Finally, it is easy to see how transitivity fails even inside a simple mathematical model.

Example 1.2.1. *Suppose there are* 3 *individuals who need to make a majority decision on which alternative to choose between* $\{x, y, z\}$. *They have the following preferences:*

$$x \succ_1 z \succ_1 y$$

$$y \succ_2 x \succ_2 z$$

$$z \succ_3 y \succ_3 x$$

Then, if the preferences are aggregated according to the majority rule, we have:

$$x \succ_{1,2} z$$

$$z \succ_{1,3} y$$

$$y \succ_{2,3} x$$

Transitivity fails a group decision. This is the famous "Condorcet's paradox of voting". One may notice that an essential feature of this example is that each agent has preferences radically different from others. Namely, each one has a different worst alternative. If this hypothesis is relaxed, group transitivity is possible. However, this example simply shows how transitivity is easy to fail.

Definition 1.2.4. A preference relation $\subseteq$ on X is represented by an **utility function** $u : X \to \mathbb{R}$ if:

$$x \subseteq y \iff u(x) \geq u(y)$$

Then, if u represents $\subseteq$, we have:

$$C^*(B, \subseteq) \equiv \Big\{x : x \in \arg\max u(x)\Big\}$$

This means that rational preferences can be transformed into an optimization problem.

Theorem 1.2.4. *A preference can be represented by a utility function only if it is rational.*

Proof. Suppose $\succeq$ on X is represented by u. For any $x, y \in X$, then, we can have $u(x) \geq u(y)$ or $u(y) \geq u(x)$, so $\succeq$ is complete. For $x, y, z \in X$, suppose $x \succeq y$ and $y \succeq z$. Then, we have:

$$u(x) \geq u(y) \geq u(z)$$

So $x \succeq z$. $\square$

Utility functions are extremely useful in describing rational decision-making because they are much easier to work with. However, it is not true that any rational preference can be represented by a utility function. One necessary condition is that the set of alternatives is finite.

Theorem 1.2.5. *A rational preference on a finite set of alternatives can be represented by a utility function.*

Proof. Suppose $n > 0$, where $n = |X|$, namely the number of alternatives in X. We can construct a set $\{X_i\}$ as follows.

- $X_0 = X$
- $X_1 = X_0 - C^*(X_0, \succeq)$
- $X_{i+1} = X_i - C^*(X_i, \succeq)$,

Define $X_{m+1} = \emptyset$ for some $m \leq n$. Let $u(x) = m - i$ if $x \in C^*(X_i, \succeq)$. Suppose $x \succeq y$, and $x \in C^*(X_i, \succeq)$. Then $u(x) = m - i$ and $u(y) \leq m - i$. We can do it for all n-elements in X. $\square$

If the set of alternatives is not countable, more conditions are needed. A famous example is that of **lexicographic** preferences.

Example 1.2.2. *Define $X = [0,1] \times [0,1]$. Then, we have $(x_1, y_1) \succeq (x_2, y_2)$ if:*

- *$x_1 > x_2$ or*
- *$x_1 = x_2$ and $y_1 > y_2$*

However, $\succeq$ has no utility representation. To see this, suppose $u(x)$ exists. Then, for every x_i, we can pick a rational number such that $u(x_1, 1) > r(x_1) > u(x_2, 2) > r(x_2) > u(x_1, 1)$ (because of the property that between any 2 real numbers, there is a rational number). This means that $r(x_1) > r(x_2)$ because $>$ is transitive. Therefore, we can construct a function $r(\cdot)$ such that:

$$r : [0,1] \times [0,1] \to \mathbb{Q}$$

But this is impossible because $[0,1]$ is an uncountable set, whereas $\mathbb{Q}$, the set of rational numbers, is countable.

The utility function ranks the alternatives but does not say anything about the relative ratio between those. This principle, initially stated by Vilfredo Pareto and then accepted by all economists, means that utility functions are unique up to any monotonic transformations. Then, we have the following theorem:

Theorem 1.2.6. *If u represents $\succeq$ and $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\psi \circ u$ represents $\succeq$.*

Proof. By definition of utility representation:

$$x \succeq y \iff u(x) \geq u(y)$$

By definition of strictly increasing function, we have:

$$\psi(x) > \psi(y) \quad \forall x, y \in \mathbb{R}$$

Combining together the 2 definitions, we have:

$$\psi \circ u(x) \geq f \circ u(y)$$

$\square$

1.2.2 Choice-Based Approach

Preferences cannot be measured or observed. A different approach looks at what people actually do, i.e., their choices.

Choices can actually be observed. So, the problem is that of studying how they are consistent or not. The primitive datum is a choice structure $(\mathcal{B}, C)$, where:

- $\mathcal{B} \subset 2^X \setminus \emptyset$, that is $\mathcal{B}$ is the power set (minus the empty set) of all possible feasible sets
- $C : \mathcal{B} \to 2^X \setminus \emptyset$: this is **choice rule**, which is a correspondence
- $C(B) \subseteq B$, $\forall B \in \mathcal{B}$.

An example is: given $X = \{x, y, z\}$, and let $\mathcal{B} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$. Then, one possible choice structure is $(\mathcal{B}, C)$ is:

$$C(\{x, y\}) = \{x\}$$
$$C(\{y, z)\} = \{y\}$$
$$C(\{x, z)\} = \{x, z\}$$

In order to make a choice analyzable, we need a definition of "consistency". This is given by the following.

Definition 1.2.5. A choice structure $(\mathcal{B}, C)$ satisfies the **weak axiom of revealed preferences** (WARP) if and only if:

- for all sets $A, B \in \mathcal{B}$, $x \in C(A)$ and $y \in A$.
- if $y \in C(B)$ and $x \in B$, then $x \in C(B)$.

Example 1.2.3. *Take $X = \{x, y, z\}$ and $\mathcal{B}\{\{x, y\}, \{x, y, z\}, \{x, z\}\}$. And the choice structure is: $(\mathcal{B}, C)$, where $C = (\{x, y\}) = \{x, y\}$. Suppose $y \in C = (\{x, y, z\})$. Then C satisfies WARP only if $C(\{x, y, z\}) = \{x, y\}$.*

In other words, this simply means that if x is chosen in a bundle where y is available, a consistent choice cannot be to choose only y (in a different situation) where x is available. In this second case, x and y must be chosen together.

1.2.3 The relationship between Preference relations and Choice rules

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Notice that the choice-based approach and the preference-based approach are not the same. In particular, the second makes it possible to represent preferences through a utility function and, therefore, to transform the problem into a utility maximization problem. However, preferences are just a "psychological" datum that cannot be observed in reality, whereas choices can, at least in principle.

Besides, the fact that these two approaches are different is not a real issue since they can be linked together. In other words, the choice structure generated by $\succeq$, that is, $C^*(B, \succeq)$ satisfies the weak axiom of real preferences.

Theorem 1.2.7. *Suppose $\succeq$ is rational. Then, the choice structure generated by $\succeq$, $C^*(B, \succeq)$ satisfies WARP.*

Proof. Suppose $x, y \in B$, and $x \in C^*(B, \succeq)$. This implies $x \succeq y$. Consider another feasible set, B', such that $x, y \in B'$. Suppose $y \in C^*(B', \succeq)$. Then, for any $z \in B'$, we have $y \succeq z$. Since $x \succeq y$ and $y \succeq z$, then, by transitivity, $x \succeq z$ and $x \in C^*(B', \succeq)$. $\square$

We have seen that if $\succeq$ is rational, then $C^*(B, \succeq)$ satisfies the weak axiom. What about the other way around? Suppose we observe a bundle of decision maker's choices that satisfies the weak axiom. Can we find preferences that are rational, or that **rationalize** the WARP-consistent choice?

Definition 1.2.6. A choice structure $(\mathcal{B}, C)$ is said to be **strongly rationalized** by a rational preference $\succeq$ if:

$$C(B) = C^*(B, \succeq), \quad \forall B \in \mathcal{B}$$

Example 1.2.4. *Suppose $X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{y, z\}\}$ and $C(\{x, y\} = \{x, y\}$, $C(\{y, z\}) = \{y\}$. This can be strongly rationalized by the following preference:*

$$x \sim y \succ z$$

Definition 1.2.7. A choice structure $(\mathcal{B}, C)$ is said to be **weakly rationalized** by a rational preference $\succeq$ if:

$$C(B) \subseteq C^*(B, \succeq), \quad \forall B \in \mathcal{B}$$

Notice that this is a weaker requirement than the previous one. Furthermore, any preference that makes the individual indifferent between any element of X will weakly rationalize any choice behavior.

However, not all the choices satisfying WARP are strongly rationalizable by rational preferences. See the following example (Mas-Colell, Whinston, and Green 1995, p. 13)

Example 1.2.5. *Suppose $X = \{x, y, z\}$ and $\mathcal{B} = \{\{x, y\}, \{x, z\}, \{y, z\}\}$, $C(\{x, y\}) = \{x\}$, $C = (\{y, z\}) = \{y\}$ and $C(\{x, z\}) = \{z\}$. This choice structure satisfies WARP. But it is not rationalizable by any rational preference. Indeed, by $C(\{x, y\})$, $x \succ y$. By $C(\{y, z\})$, $y \succ z$. But $\succ$ is transitive, then $x \succ z$, which contradicts $C(\{x, z\}) = \{z\}$.*

It can also be the case that if a choice structure can be rationalized by $\succeq$, still, it is not unique.

Example 1.2.6. *Suppose* $X = \{x, y, z\}$*, and* $\mathcal{B} = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$*,* $C(\{x, y\}) = \{x\}$*,* $C = (\{y, z\}) = \{x\}$ *and* $C(\{x, y, z\}) = \{x\}$*. Then, again, WARP is satisfied, and we have* $x \succ y$ *and* $x \succ z$*, but we don't know anything about* y *and* z*.*

If $\mathcal{B}$ contains enough subsets of X, and if $(\mathcal{B}, C)$ satisfies the weak axiom, then there exists a rational preference that rationalizes $C(\cdot)$, relative to $\mathcal{B}$.

Theorem 1.2.8. *If* $(\mathcal{B}, C)$ *is a choice structure that:*

1. *satisfies WARP*
2. *contains all the subsets of* X *up to three elements,*

Then, there is a rational preference relation $\succeq$ *that rationalizes* $C(\cdot)$ *relative* $\mathcal{B}$*, i.e.:*

$$C(B) = C^*(B, \succeq)$$

Furthermore, this rational preference is the unique doing so.

Proof. We want to show that 1) $\succeq$ is rational, that 2) $\succeq$ rationalizes $C(\dot{})$, and that 3) it is the unique preference to rationalize $C(\cdot)$.

To see 1) $\succeq$ must be complete and transitive. Complete means that either $x \succeq y$ or $y \succeq x$. Since $\mathcal{B}$ comprises all the subsets of X up to three elements, then $B = \{x, y\} \subseteq \mathcal{B}$. Then, or $x \in C(\{x, y\})$ or $y \in C(\{x, y\})$ or both. In any case, $x \succeq y$ or $y \succeq x$.

Let's see transitivity now. We want to show that wherever x, y, z, $x \succeq y$ and $y \succeq z$, this implies $x \succeq z$. $C(\{x, y, z\}) \neq \emptyset$, since $\{x, y, z\} \in \mathcal{B}$. Then, we have three cases:

1. $x \in C(\{x, y, z\})$. This trivially implies transitivity.
2. $y \in C(\{x, y, z\})$. This means $y \succeq x$. But we have also $x \succeq y$. So it exists a $B = \{x, y\}$ such that $x \in C(B)$, and by WARP, we have $x \in C(\{x, y, z\})$.
3. $z \in C(\{x, y, x\})$. $y \succeq z$ means that there is a $B = \{y, z\}$ and $y \in C(B)$. By WARP, $y \in C(\{x, y, z\})$. And then, by the argument above, $x \in C(\{x, y, z\})$.

Let's see 2).

$C(B) = C^*(B, \succeq)$ means:

$$C(B) \subseteq C^*(B, \succeq)$$
$$C^*(B, \succeq) \subseteq C(B)$$

To see the first subset, notice that $x \in C(B)$ reveals $x \succeq y$. So $x \in C^*(B, \succeq)$, and $C(B) \subseteq C^*(B, \succeq)$.

Let's show $C(B) \subseteq C^*(B, \succeq)$. That is, if $x \in C^*(B, \succeq)$, then $x \in C(B)$. $x \in C^*(B, \succeq)$ implies that $x \succeq y$. Pick $y \in C(B)$. Since $x \succeq y$, then there exists $B \in \mathcal{B}$ such that $B = \{x, y\}$, $y \in C(B)$, implies $x \in C(B)$ by WARP. So $C^*(B, \succeq) \subseteq C(B)$, for all $B \in \mathcal{B}$.

Finally, for uniqueness, since all two-element subsets of X are contained in $\mathcal{B}$, then $C(\cdot)$ completely exhausts any pairwise preference relations on X for any rationalizing $\succeq$. $\square$

1.3 Consumer Choice

The basic decision unit of microeconomic theory is the consumer. Assuming that the consumer is part of a market economy, namely a situation where goods and services are available for purchase at known prices, we can define the fundamentals as follows.

The consumer is the rational decision-maker. Her scope is to purchase or sell goods $l = 1, \ldots, n$. The set of all feasible commodities bundles is given by X, now called **consumption set**. This, together with the **Walrasian budget set**, represents the constraints of the consumer problem. Furthermore, the choice rule (seen in the context of the choice-based approach to individual decision-making), is called **Walrasian demand function**.

1.3.1 The Consumption Set

Usually, it is assumed that $X = \mathbb{R}^L_+$, so a typical consumption bundle is given by:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix} \geq 0$$

The consumption set has the following features:

- Unbounded: each consumer is too small to exhaust all the goods
- Perfectly divisible (since $X \subseteq \mathbb{R}^L_+$). This does not exclude that there are discrete goods that are not indivisible. In that case, we can accommodate the utility function on consequence.
- It is a closed set.
- It is a convex set: that is $(x, y) \in X$, then $\alpha x + (1-\alpha)y \in X, \quad \forall x, y \in X, \alpha \in [0, 1]$. Convexity is both mathematically and economically important.

1.3.2 Competitive Budget Sets

The Consumption set represents a sort of physical constraint. In addition to it, we have an economic constraint. How many commodity bundles a consumer can afford. This can be formalized by introducing two assumptions:

- The L commodities are traded in the market at dollar prices "publicly quoted" (or, said otherwise, markets are complete). These prices then can be represented by a **price vector**:

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$$

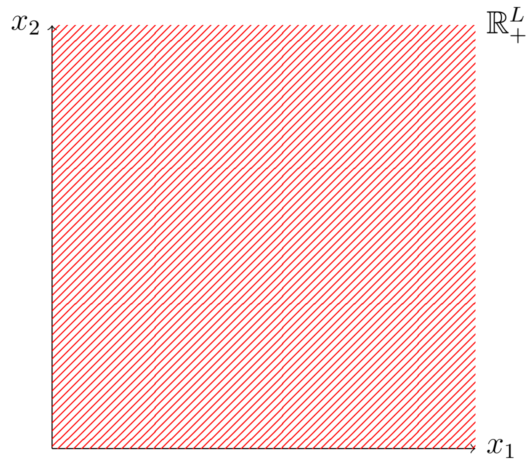


Figure 1.1: An example of Consumption Set

Prices can be negative, meaning that a consumer is giving away a commodity or she is being paid to consume. In any case, for most of the general uses, it is assumed the prices are strictly positive.

- prices are beyond the control of one consumer. This is the **price-taking** assumption.

Furthermore, prices are linear. This means that every unit of good k has the same price, p_k.

If a consumption bundle is affordable, it depends on the prices and the consumer's wealth level. Then, a consumption bundle is affordable if its total costs do not exceed the consumer's wealth level:

$$\mathbf{p} \cdot \mathbf{x} = p_1 x_1 + \cdots + p_L x_L \leq w$$

Then, we can define the **Walrasian budget set**.

Definition 1.3.1. The Walrasian, or competitive budget set

$$B_{p,w} = \left\{ x \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} \leq w \right\}$$

is the set of all feasible consumption bundles for the consumer, at prices p and wealth w.

The set $\{x \in \mathbb{R}^L_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$ is called **budget hyperplane**, and it is the upper boundary of the budget set. The slope of the budget hyperplane $-\frac{p_1}{p_2}$ is the rate of exchange between the two commodities. If the price of a commodity 2 decreases, then with the same price, the consumer can buy more units of it. Then, the budget set becomes larger.

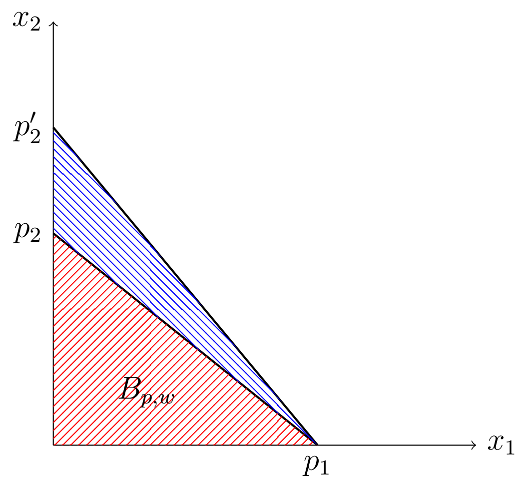


Figure 1.2: An example of Budget Set for $L = 2$. The effect of a price change

The Budget set is homogeneous of degree zero, namely if prices and wealth are both multiplied by $\alpha > 0$, the budget set does not change.

Furthermore, the budget set is convex. This means that if $x, y \in B_{p,w}$, then any convex combination of x, y is still in $B_{p,w}$.

The last result is extremely important for consumer theory.

Proposition 1. *If $p \gg 0$, then $B_{p,w}$ is compact.*

Proof. We must show that it is closed and bounded (Heine-Borel Theorem).

To see that it is closed, take any sequence $\{x_n\} \in B_{p,w}$. This means $p \cdot x_n \leq w$. By definition of closed set, then x s.t $\{x_n\} \to x$ is also in $B_{p,w}$. Notice that weak inequality is preserved under limit operators. This means that $p \cdot x \leq w$.

Let's see that $B_{p,w}$ is bounded. Let's define $P = \min p_j > 0$. Then, we can write:

$$w \geq p \cdot x \geq \sum_j p_j x_j \geq P(\sum_j x_j)$$

Then $\sum_j x \leq \frac{w}{P}$. $\square$

1.3.3 Demand Correnspondences

The **demand correspondence** (or Walrasian demand) $x(p, w)$ assigns a set of chosen consumption bundles for each price-wealth pair (p, w). This can be a correspondence (a set-valued function) or a single-valued function. In the last case, we speak of **demand function**. In any case, $x(p, w) \subseteq B_{p,w}$.

There are two main assumptions regarding the Walrasian demand correspondence: the **homogeneity of degree zero**, and the **Walras' Law**

Definition 1.3.2. The Walrasian demand correnspondence (function) $x(p, w)$ is homogeneous of degree zero if $x(\alpha p, \alpha w) = \alpha^0 x(p, w) = x(p, w)$, for any $\alpha > 0$.

This means that if both prices and wealth change in the same proportion, then the individual's choice does not change. This because the consumption bundle does not change $B_{p,w} = B_{\alpha p,\alpha w}$

Example 1.3.1. *Consider the following demand function:*

$$\mathbf{x}(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \frac{\alpha p_2}{p_1+p_2} \cdot \frac{w}{p_1} \\ \frac{\beta p_1}{p_1+p_2} \cdot \frac{w}{p_2} \end{bmatrix}$$

This satisfies Walras's law if $\mathbf{x} \cdot \mathbf{p} = w$. *Then:*

$$\mathbf{p}.\mathbf{x} = \left[p_1 \cdot \frac{\alpha p_2}{p_1+p_2} \cdot \frac{w}{p_1} + p_2 \cdot \frac{\alpha p_2}{p_1+p_2} \cdot \frac{w}{p_1}\right] =$$

$$\left[\frac{\alpha p_2}{p_1+p_2} + \frac{\alpha p_2}{p_1+p_2}\right] \cdot w = \left[\frac{p_1+p_2}{p_1+p_2}\right] w = w \ \ \textit{if } \alpha = \beta = 1$$

Definition 1.3.3. The Walrasian demand correspondence (function) $x(p,w)$ satisfies Walras' Law, if, for every $p \gg 0$, and $w > 0$, we have $\mathbf{p} \cdot \mathbf{x} = w$, for all $x \in x(p,w)$.

Roughly speaking, this means that the consumer spends all his wealth (over her lifetime)

Example 1.3.2. *Consider the demand function of above:*

$$\mathbf{x}(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \end{bmatrix} = \begin{bmatrix} \frac{\alpha p_2}{p_1+p_2} \cdot \frac{w}{p_1} \\ \frac{\beta p_1}{p_1+p_2} \cdot \frac{w}{p_2} \end{bmatrix}$$

This is homogeneous of degree zero for all $\alpha, \beta > 0$. *Indeed:*

$$x_1(\alpha p, \alpha w) = \frac{\alpha^2 p_2}{\alpha p_1 + \alpha p_2} \cdot \frac{\alpha w}{\alpha p_1} = \frac{\alpha p_2}{\alpha(p_1+p_2)} \cdot \frac{\alpha w}{p_1} = \frac{\alpha p_2}{p_1+p_2} \cdot \frac{w}{p_1} = x_1(p,w)$$

$$x_2(\beta p, \beta w) = \frac{\beta^2 p_2}{\beta p_1 + \beta p_2} \cdot \frac{\beta w}{\beta p_1} = \frac{\beta p_2}{\beta(p_1+p_2)} \cdot \frac{\beta w}{p_1} = \frac{\beta p_2}{p_1+p_2} \cdot \frac{w}{p_1} = x_2(p,w)$$

1.3.4 Some Comparative Statics

We are interested in studying how the choice of consumers changes with changes in prices and wealth.

There are two types of effects, the **wealth effects** and the **price effects**. As the name suggests, the first examines the effect on the demand of a change in wealth and the second of a change in prices. This change is represented by the partial derivatives:

$$\frac{\partial x_i(p,w)}{\partial w} \text{ and } \frac{\partial x_i(p,w)}{\partial p_i}$$

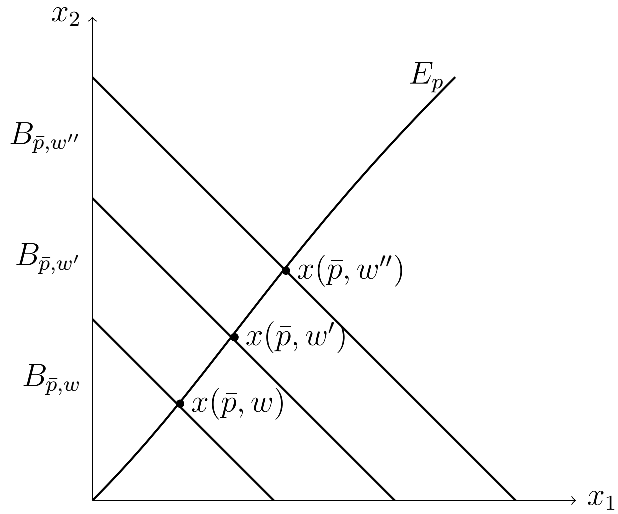


Figure 1.3: The wealth expansion path at price $\bar{p}$

Wealth effects

Fixing the price $\bar{p}$, the function of wealth $x(\bar{p},w)$ is called **Engel's curve**, and its image represents the wealth expansion path. For any combination (p,w), the derivative $\frac{\partial x_i(p,w)}{\partial w}$ is the wealth effect for the demand of the i-th good.

Then, we have the following characterization of goods:

- A good i is called **normal** at (p,w) if:

$$\frac{\partial x_i(p,w)}{\partial w} \geq 0$$

  That is if the demand is non-decreasing with wealth. Assuming that the standard demand for a good is increasing with wealth, namely, if the consumers' wealth increases, the demand for that good increases too, or at least remains the same, a normal good does not display the opposite effect.

- A good is called **inferior** at (p,w) if:

$$\frac{\partial x_i(p,w)}{\partial w} \leq 0$$

  Namely, if the consumer's wealth increases and the consumption decreases. A standard example is the substitution of some kind of cheap products, say cheap food, with better quality ones as the consumer's income increases.

The wealth effects can be represented by the following $(1 \times L)$ matrix:

$$D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L$$

Price effects

Price effects determine how consumption levels of the various commodities change as prices change.

More generally, the derivative $\frac{x_i(p,w)}{\partial p_k}$ is the price effect of the change of the price of good k on the demand for good l $(l = 1\ldots, k, \ldots L)$. Then we have:

- A good is said to be **ordinary** if:

$$\frac{\partial x_j(p,w)}{\partial p_j} < 0$$

  Namely, if the price increases, then the demand decreases.

- A good is said to be **Giffen**[1], if:

$$\frac{\partial x_j(p,w)}{\partial p_j} > 0$$

  Namely, if the price increases but also the demand increases. The standard example (thankfully outdated!) is that of potatoes and meat. Imagine a poor consumer with low wealth whose diet is made up of potatoes six days a week and meat one day a week. If the price of potatoes increases, then she cannot afford any more meat on the seventh day. So, the overall demand for potatoes increases, even if their price has increased too.

A useful way of representing consumer demand for each level of (p, w) is the **offer curve**.

The price effects can be represented by the following $(L \times L)$ matrix:

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} \ldots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} \ldots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$$

Furthermore, we say that:

[1] Alfred Marshall attributed this idea to the XIXth century Scottish economist Robert Giffen

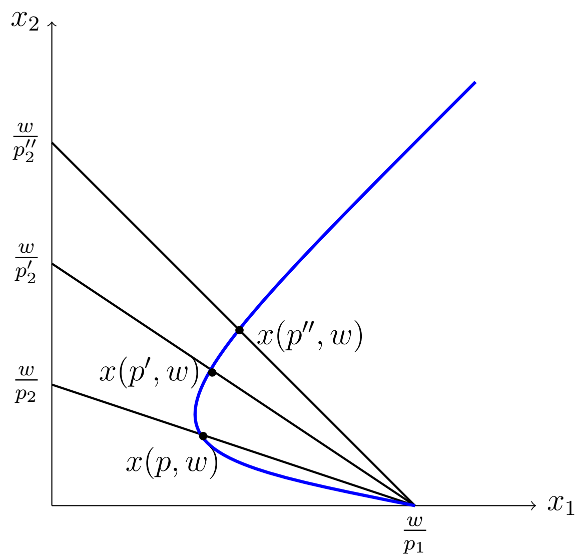


Figure 1.4: An example of offer curve, where p_2 changes ($p_2'' < p_2' < p_2$)

- A good k is a (gross) **substitute** for good l if:

$$\frac{\partial x_l(p,w)}{\partial p_k} > 0$$

  Namely, if the demand of l increases when the price of k increases.

- A good k is a (gross) **complement** for good l if:

$$\frac{\partial x_l(p,w)}{\partial p_k} < 0$$

  Namely, the demand of l decreases when the price of k increases.

Implications of homogeneity of degree zero and Walras' Law

Both Homogeneity of degree zero and Walras' Law imply certain restrictions for price and wealth effects of consumer demand.

Let's start with the implications of homogeneity of degree zero.

Proposition 2. *If the Walrasian demand function $x(p,w)$ is homogenous of degree zero, then for all p and w, we have:*

$$\sum_{i=1}^{n} \frac{\partial x(p,w)}{\partial p_i} \cdot p_k + \frac{\partial x(p,w)}{\partial w} \cdot w = 0 \quad \textit{for } i = 1 \ldots, n$$

Proof. By homogeneity of DZ, we can write:

$$x(\alpha p, \alpha w) - x(p,w) = 0, \quad \alpha > 0$$

Differentiating this with respect to α (and evaluating at $\alpha = 1$) we have:

$$\sum_{i=1}^{n} \frac{\partial x(p,w)}{\partial p_i} \cdot p_k + \frac{\partial x(p,w)}{\partial w} \cdot w = 0 \tag{1.1}$$

□

Notice that this is a special case of Euler's theorem, according to which if f is differentiable, homogeneous of degree r and differentiable, then we have:

$$\sum_{i=1}^{n} \frac{\partial f(x_1,\ldots,x_n)}{\partial x_i} \cdot x_i = rf(x_1,\ldots,x_n)$$

This means that homogeneity of degree zero implies that price and wealth derivatives for any good l, when weighted by these prices and wealth, sum to zero.

This can also be restated in terms of **elasticities** by multiplying each term by $\frac{1}{x_l(p,w)}$. Then we can write:

$$\varepsilon_{l,p_k}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} \cdot \frac{p_k}{x_l(p,w)}$$
$$\varepsilon_{l,w}(p,w) = \frac{\partial x_l(p,w)}{\partial w} \cdot \frac{p_k}{x_l(p,w)}$$

These elasticities give the percentage change in demand for good l per marginal percentage change in the price of good k or wealth. Then, using elasticities, we can rewrite (1) as:

$$\sum_{i=1}^{n} \varepsilon_{l,p_k}(p,w) + \varepsilon_{l,w}(p,w) = 0 \quad \text{for } l = 1,\ldots,n \tag{1.2}$$

Walras's Law has two implications for the price and wealth effects of demand.

Proposition 3. *If $x(p,w)$ satisfies Walras' Law, then, for all p and w, we have:*

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \quad \forall k = 1,\ldots,L \tag{1.3}$$

Proof. By Walras' Law, we have:

$$\mathbf{p} \cdot \mathbf{x} = w$$

This can be written as:

$$\Big[p_1 x_1(p,w) + \cdots + p_L x_L(p,w)\Big] = w$$

Differentiating with respect to prices, we have:

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \quad \forall k = 1,\ldots,L$$

□

This is called **Cournot Aggregation**. This means that the total expenditure cannot change in response to a change in prices.

Proposition 4. *If $x(p,w)$ satisfies Walras' Law, then, for all p and w, we have:*

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial w} = 1 \quad \forall k = 1, \ldots, L \tag{1.4}$$

Proof. As above, by Walras's Law, we have:

$$\mathbf{p} \cdot \mathbf{x} = w$$

Differentiating with respect to w, we have:

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial w} = 1$$

□

This is called **Engel aggregation**. This means that the total expenditure must change by an amount equal to any wealth change.

Finally, we can easily pass from (3) e (4) to the elasticity formulas (5) and (6):

Let's see (4):

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0$$

Multiply by $\frac{p_k}{w}$. Then we have:

$$\sum_{i=1}^{n} p_i \frac{\partial x_l(p,w)}{\partial p_k} \frac{p_k}{w} + \frac{p_k x_k(p,w)}{w} = 0$$

Multiplying both elements by $\frac{x_l(p,w)}{x_l(p,w)} = 1$, and rearranging, we have:

$$\sum_{i=1}^{n} \frac{p_l x_l(p,w)}{w} \cdot \frac{\partial x_l(p,w)}{\partial p_k} \cdot \frac{p_k}{x_l(p,w)} + \frac{p_k x_k(p,w)}{w} = 0 \tag{1.5}$$

Where $\varepsilon_{l,p_k}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} \cdot \frac{p_k}{x_l(p,w)}$ and $\frac{p_l x_l(p,w)}{w}$ is the budget share of the consumer's expenditure on good l given prices p and wealth w.

Let's see (4):

$$\sum_{i=1}^{L} p_i \frac{\partial x_l(p,w)}{\partial w} = 1$$

Multiplying both sides for $\frac{x_l(p,w)}{x_l(p,w)} \cdot \frac{w}{w}$ and rearranging, we have:

$$\sum_{i=1}^{L} \frac{p_l x_l(p,w)}{w} \cdot \frac{\partial x_l(p,w)}{\partial w} \cdot \frac{w}{x_l(p,w)} = 1 \tag{1.6}$$

Where $\frac{\partial x_l(p,w)}{\partial w}$ is the elasticity of demand with respect to wealth.

Chapter 2

Classical Demand Theory

2.1 Introduction

The traditional approach to consumer behavior is to assume that each consumer has well-defined preferences over all the alternative bundles so that she tries to select her preferred bundles among those available. Furthermore, we have seen if preferences are rational and the set of alternatives is finite, these can be represented by a utility function.

These results have been generalized in the "neo-classical" theory of consumer demand, where demand solves the utility maximization problem, as well as its dual, the expenditure minimization problem.[1]

2.2 Consumer preferences

Recall that the consumer chooses among different alternatives in the consumption bundle $X \subseteq \mathbb{R}^L_+$. This choice is reflected in, say, the preference of x over y, and we write $x \succeq y$. $\succeq$ is rational if it is complete and transitive.

An assumption that is often made is that large quantities of goods are preferred to lesser quantities. This idea is formalized in the notion of **monotonicity**. In particular, we have three "versions" of it, from the weakest one to the strongest one.

Definition 2.2.1. A preference $\succeq$ of X is:

[1] The origins of this theory date back to the end of the XIXth Century, when some scholars, such as Alfred Marshall, Frances Ysidro Edgeworth, and Vilfredo Pareto, developed some models of demand theory, and, especially Pareto, started to investigate the mathematical properties of Utility functions (Edgeworth 1881; Marshall 1920; Pareto 2014(1906)) These results have been generalized in a mathematical fashion after the 1930s in some works as Hicks and Allen 1934; Hicks 1939 and Samuelson 1947. Finally, these have been rigorously proved from the 1950s onwards, together with the development of strong mathematical techniques to address these problems (the first comprehensive example is: Debreu 1959)

- **Locally non-satiable** if any open neighborhood of x, i.e. $\|x - y\| \leq \epsilon$ contains a bundle $y \in X$ such that $y \succ x$
- **monotone** if $x_i > y_i$ for $i = 1, \ldots, n$ implies $x \succ y$ for any x, y
- **strongly monotone** if $x \geq y (x_i \geq y_i, \forall i)$ implies $x \succ y$

Strong monotonicity just requires greater or equal to, for all x_i, y_i. Monotonicity instead requires strict inequality. Therefore, preference relations that are strongly monotone are also monotone, the opposite is not always true: strong monotonicity is a stronger assumption.

Example 2.2.1. *Consider the bundles $x = (1, 1)$ and $y = (1, 2)$. If $\succeq$ is strongly monotone, then $y \succ x$. However, if $\succeq$ is only monotone, then we cannot say $y \succ x$, since $x_1 = y_1$.*

Local non-satiation is the weakest condition. This means that for every bundle x, we can always define a $\epsilon-$neighborhood such that $y' \in N_\epsilon \succ x$.

Example 2.2.2. *The classical example of preferences which does not satisfy local non-satiation is given by the utility function (however, notice that we have not already introduced the notion of utility function):*

$$u(x) = -\|x - \alpha\|$$

A real-life example is given by voter's utility functions. Each voter has a preferred candidate. Therefore, her utility is given by the euclidean distance from the position of the candidate.

Local non-satiation rules out the possibility of thick indifference curves (see below).

Given $\succeq$ and a consumption bundle x, we can define three sets. The **upper contour set** of x (or NBT(x)) is the set of all bundles that are at least as good as x, namely:

$$\Big\{x \in X : y \succeq x\Big\}$$

Similarly, the **lower contour set** (or NWT(x)) of x is the set of bundles that x is at least as good as, namely:

$$\Big\{y \in X : x \succeq y\Big\}$$

These sets are **closed**, and their intersection is the set of all bundles that are indifferent to x, namely:

$$\Big\{y \in X : y \sim x\Big\}$$

This last set is also called **indifference curve**. Graphically, we have:

A second assumption for preferences is that of **convexity**. This concerns the trade-offs that the consumer is willing to make between different goods. In particular, convexity can be seen as an "inclination for diversity": one consumer prefers a combination of two goods instead of allocating all her income to one good.

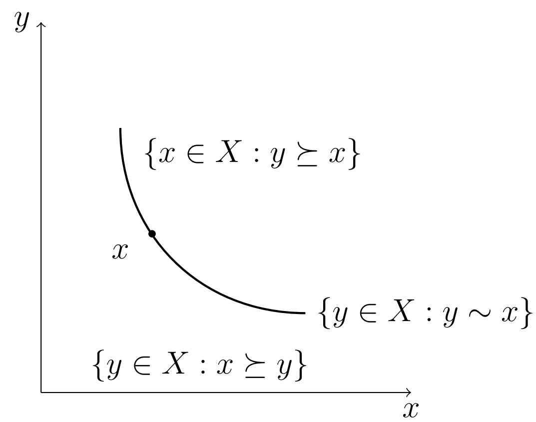


Figure 2.1: Upper contour set, Lower counter set, and indifference curve

Definition 2.2.2. The preference $\succeq$ on a convex choice set X is **convex** if and only if $x \succeq y$ and $x' \succeq y$ implies that for any $\alpha \in (0,1)$, $\alpha x + (1-\alpha)x' \succeq y$. $\succeq$ is **strictly convex** if and only if $x \succeq y$ and $x' \succeq y$ and $x \neq y$, imply that for any $\alpha \in (0,1)$, $\alpha x + (1-\alpha)x' \succ y$.

A way to interpret convexity in economics is in terms of **marginal rate of substitution**, that is, the slope of the indifference curves. These are diminishing, and this relates to the intuitive idea of diminishing marginal utility, namely, given an initial bundle (x, y), it takes increasingly larger quantities of x to compensate the losses to y in order to keep total utility unchanged.

Furthermore, a theorem links convex preferences and **quasi-concave utility functions**.[2]

Theorem 2.2.1. *Suppose a rational and continuous preference $\succeq$ is represented by u. It is (strictly) convex if and only if $u(\cdot)$ is (strictly) quasi-concave.*

Proof. We see only QC $\Rightarrow$ convex preferences (the other direction is similar). Define $x \succeq x'$ and $x' \succeq x'$ (by completeness). Then we can write:

$$\alpha x + (1-\alpha)x' \succeq x'$$

Since $\succeq$ can be represented by u, we have:

$$u(\alpha x + (1-\alpha)x') \geq u(x')$$

[2]Quasi-concave functions are a class of functions that maintain certain properties of convex and concave functions but are less demanding. In particular, Quasi-concavity is always preserved under monotonic transformations (and this is not true for concave functions.

Definition 2.2.3. Let $f : E \subseteq \mathbb{R}^n$ be convex. A function $f : E \to \mathbb{R}$ is **quasi-concave** if:

$$f(\lambda x + (1-\lambda)x') \geq \min\{f(x'), f(x)\} \;\; \forall x, x' \in E, \forall \lambda \in [0,1]$$

If the inequality is strict, for all $\lambda \in (0,1)$, then we have **strict quasi-concavity**.

Then, since $x \succeq x'$ and $x' \succeq x'$, we can write $u(x') = \min\{u(x), u(x')\}$, and:

$$u(\alpha x + (1-\alpha)x') \geq u(x')$$

$\square$

As we will see more in detail below, if $u(\cdot)$ is quasi-concave, then $x(p,w)$ is a convex set, and if $u(\cdot)$ is strictly quasi-concave, then $x(p,w)$ is a singleton (or empty).

An important assumption (needed, as we will see, to ensure the existence of utility functions) is **continuity**.

Definition 2.2.4. A preference $\succeq$ on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x_n, y_n)\}_{n=1}^{\infty}$, with $x_n \succeq y_n$, for all n, and $x_n \to x, y_n \to y$, then $x \succeq y$.

Example 2.2.3. *An example of non-continuous preferences are **lexicographic preferences**. Indeed, take $x = (\frac{1}{n}, 0)$ and $y = (0,1)$. Then, we have $x \succ y$. But, taking the limit, we have $y \succ x$. (Further, recall that lexicographic preferences cannot be represented by a utility function because they are a map from an uncountable set to a countable one).*

We can link the continuity of preferences and upper(lower) contour sets in the following way.

Lemma 2.2.2. *The following statements are equivalent:*

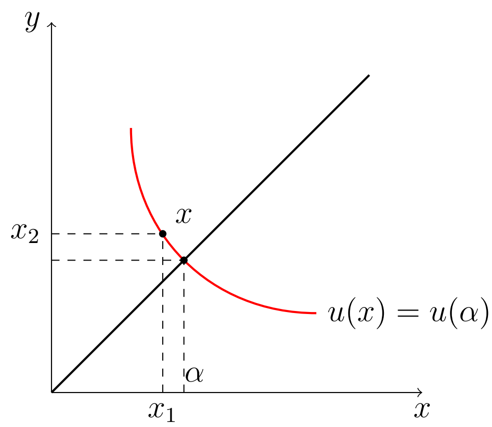
- *A preference is continuous*
- *Upper and lower contour sets are closed sets (therefore, their complementary, the set of bundles that are strictly best or strictly worse than x, are open)*
- *If $x \succeq y$, then a neighborhood $N_\epsilon(x)$, with $\epsilon > 0$ exists, such that $x' \in y$, for all $x' \in N_\epsilon(x)$.*

If the set of alternatives is countable, we can represent those by means of a utility function. If the set is not countable (like $X = \mathbb{R}^L$), but preferences are continuous, we can still represent preferences using a utility function. Furthermore, the utility function is continuous, so many desirable properties, starting with differentiability, apply. The following result shows this.

Theorem 2.2.3. *(Monotone Representation Theorem) A continuous rational $\succeq$ on $\mathbb{R}^L_+$ that is monotone is representable by a continuous function $u(x)$.*

Proof. To get the intuition, we can see the figure (when $L = 2$):

Let $x \in \mathbb{R}^2_+$. We can define two subsets, $A^+(x) = \{\alpha \in \mathbb{R} : \alpha e \succeq x\}$ and $A^-(x) = \{\alpha \in \mathbb{R} : \alpha e \preceq x\}$, where $e = (1, 1, \ldots, 1)$. By monotonicity both $A^+(x)$ and $A^-(x)$ are non empty. By continuity, the sets are closed (notice that this is another way of



defining upper and lower contour sets). Since $\succeq$ is rational, then it is complete, so $A^+(x) \cup A^-(x) = \{x \in X : x = \alpha e, \alpha \in \mathbb{R}_+\}$. This set is connected (namely is not the union of two separated non-empty sets). The set $A^+(x) \cap A^-(x)$ is a singleton, and we can define its element as $\alpha(x)$. Then it exists a unique $\alpha(x)$ such that $\alpha(x)e \sim x$. So we can write $\alpha(x) = u(x)$. This for what concerns the construction of the proof. Let's see now that $\succeq$ can be represented by $u(\cdot)$.

($\Rightarrow$) Suppose $u(x) \geq u(y)$. If $u(x) = u(y)$, then $x \sim \alpha(x)e \sim y$, so we have $x \sim y$, and $x \succeq y$. If $u(x) > u(y)$, then $a(x)e \succ \alpha(y)e$ (by monotonicity), and since $x \sim \alpha(x)e$ and $y \sim \alpha(y)e$, then $x \succ y$.

($\Leftarrow$) Suppose $x \succeq y$. Then $\alpha(x)e \sim x \succeq y \sim \alpha(y)e$. Hence $\alpha(x)e \succeq \alpha(y)e$ and by monotonicity $u(x) \geq u(y)$.

The last thing we need to prove is that $u(\cdot)$ is continuous. A definition of continuity for functions is that the preimage of every open subset of the co-domain must be open. In this case, it means that $u^{-1}(\alpha, \infty)$ and $u^{-1}(-\infty, \alpha)$ must be open in the domain. Since $u(\alpha, \ldots, \alpha) = \alpha$, these sets are strict contour sets. Therefore, they are open. So $u(\cdot)$ is continuous. $\square$

Notice that this theorem heavily rests upon monotonicity. In general, this assumption is not necessary for utility representations. A stronger result is the following.

Theorem 2.2.4 (Debreu's Representation Theorem)**.** *For any a and $b > a$, $a, b \in \mathbb{R}$, a continuous rational preference on a connected set $X \subseteq \mathbb{R}^L$ is represented by a continuous function $u : X \to [a, b]$.*

If a continuous function u represents a preference $\succeq$, then $\succeq$ is continuous.

2.2.1 Some consumer utility functions

In some applications, it is extremely useful to use preferences and utility functions where it is possible to deduce the entire consumer's preference relation from a single indifference set. Three examples are **homothetic** preferences, **separable utility** and **quasilinear utility**.

Let's start with homotheticity.

Definition 2.2.5. A preference $\succeq$ is homothetic if $x \succeq y$ implies that $\lambda x \succeq \lambda y$, $\forall \lambda \in \mathbb{R}_+$.

A corollary of this result is the following:

Lemma 2.2.5. *A homothetic preference implies that if $x \sim y$, then $\lambda x \sim \lambda y$,$\forall \lambda \in \mathbb{R}_+$.*

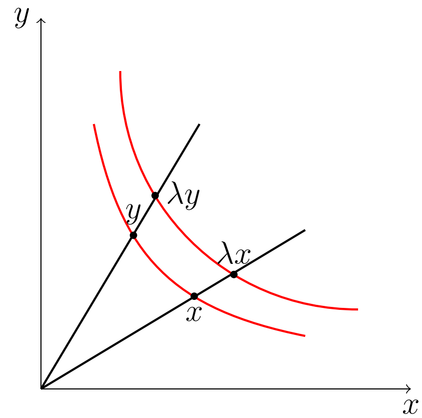


Figure 2.2: Homethetic preferences

Homothetic preferences have some nice properties, especially, as the figure suggests, the income expansion path is linear. The following result links homotheticity and homogenous utility functions.

Theorem 2.2.6. *A rational, continuous, and monotone preference is homothetic if and only if it can be represented by a continuous and homogeneous of degree one utility function.*

Proof. Recall that homogeneity of degree one means $u(\lambda x) = \lambda u(x)$. ($\Rightarrow$) For any x, there exists an α such that $x \sim \alpha$, and $u(x) = \alpha(x)$. Since the preference is homothetic, then $\lambda x = \lambda \alpha$, and $u(\lambda x) = \lambda \alpha(x) = \lambda u(x)$. ($\Leftarrow$) $x \geq y$ implies $u(x) \geq u(y)$ By homogeneity, $u(\lambda x) \geq \lambda u(x) = \lambda u(x) = u(\lambda x)$. $\square$

Example 2.2.4. *The following are examples of homothetic utility functions:*

- *the **Cobb-Douglas** utility function $u(x, y) = x^\alpha y^{1-\alpha}$.*
- *The utility function for complementary goods (**Leontief preferences**): $u(x, y) = \min\{\frac{x}{m_1}, \frac{x}{m_2}\}$ (where m_1 and m_2 are the quantities of goods x and y).*

Another utility function class involves **separable utility**. In many cases, it is convenient to assume that the decision maker's preferences are separable, namely that the preferences over different goods are independent. A standard representation for separable utility is:

$$u(x_1, \ldots, x_n) = \sum_{i=1}^{n} u(x_i) \quad i = 1, \ldots, n$$

The implicit assumption on which these class of utility functions are based is the following:

Definition 2.2.6. A preference $\succeq$ on X satisfies **double cancellation property** if, for all $x, y, z \in X$, we have that $(x_1, x_2) \succeq (y_1, y_2)$ and $(y_1, z_2) \succeq (z_1, x_2)$ implies:

$$(x_1, z_2) \succeq (z_1, y_2)$$

In words, this property states that if a bundle with (x_1, x_2) is weakly preferred to a bundle with (y_1, y_2), and a bundle with (y_1, z_2) is weakly preferred to a bundle with (z_1, x_2), then a bundle with (x_1, z_2) must be weakly preferred to a bundle with (z_1, y_2), since $x_1 \succeq y_2$ and $z_2 \succeq z_1$. This property is necessary to guarantee that $\succeq$ can be represented by separable utility functions. See the following lemma:

Lemma 2.2.7. *Suppose $\succeq$ is represented by $u(x, y)$. $u(x, y)$ is separable only if $\succeq$ satisfies double cancellation property.*

Proof. By double cancellation property and representation theorem, we have that:

$$u(x_1, x_2) \geq u(y_1, y_2)$$

and

$$u(y_2, z_2) \geq u(z_1, x_2)$$

implies:

$$u(x_1, z_2) \geq u(z_1, y_2)$$

But this is true only if $u(\cdot)$ is separable. Indeed:

$$u(x_1) + u(x_2) \geq u(y_1) + u(y_2)$$

and

$$u(y_1) + u(z_2) \geq u(z_1) + u(x_2)$$

implies

$$u(x_1) + u(z_2) \geq u(z_1) + u(y_2)$$

Because all the left-side elements are greater or equal to all the right-side elements. $\square$

A third class of utility functions are those with **quasi-linear utility**. These are used to represent those situations where the utility of one (or more) goods is linear because we are interested only in some goods. The standard form is:

$$u(x, y) = y + u(x)$$

We are interested in the utility of x. Therefore, any change of y is kept constant. Most importantly, this eliminates the income and wealth effects of price changes (see below).

Definition 2.2.7. A rational preference $\succeq$ on $X = \mathbb{R}^{N+1}$ satisfies the following properties:

- $(t, y) \succeq (t', y)$ iff $t \geq t'$, where $t, t' \in \mathbb{R}$
- for every $y, y^* \in \mathbb{R}^N$, there exists some $t \in \mathbb{R}$ such that $(0, y) \sim (t, y^*)$
- if $(t, y) \succeq (t', y')$ then for all $d \in \mathbb{R}$, $(t + d, y) \geq (t' + d, y')$

if and only if it can be represented by a quasi-linear utility function $t + v(x)$

2.3 The Utility Maximization Problem

So far, we have defined the consumer's problem as one involving preferences and their representation through a utility function. However, once we have found a utility function and a budget set as well, the problem is to determine what the actual choice of the consumer is. This is obtained by solving the following optimization problem:

$$v(p, w) = \max_{x \in \mathbb{R}^L_+} u(\mathbf{x})$$

$$\text{s.t.}$$

$$\mathbf{p} \cdot \mathbf{x} \leq w$$

The set of optimal solution, $\mathbf{x}^*(p, w)$, is called **Marshallian Demand correnspondence**.[3]

$$x^*(p, w) = \arg\max_{x \in B_{p,w}} u(x)$$

If this set is single-valued, then we have the **Marshallian demand function**. In the case of 2 goods, we have the following familiar geometrical representation:

The demand is given by the point of tangency between the indifference curve and the budget set. Intuitively, this comes from Walras' Law and Local non-satiation. The following proposition states the main properties of the demand correspondence.

Proposition 5. *Suppose that $u(\cdot)$ is a continuous utility function representing $\succeq$ on $X \subseteq \mathbb{R}^L_+$, and $\succeq$ is Locally non-satiated. Then, the* ***Marshallian correnspondence*** *satisfies the following properties:*

1. ***Homogeneity of degree zero*** *in (p, w). Namely, $x(p, w) = x(\alpha p, \alpha w)$.*
2. ***Walras' Law**: $\mathbf{p} \cdot \mathbf{x} = w$, for all $x_i \in \mathbf{x}(p, w)$.*

[3] Some (Mas-Colell, Whinston, and Green 1995, p. 51) refer to it also as **Walrasian Demand correnspondece**. In any case, the two terms can be used interchangeably. I think that the preference for one over the other rests mainly on different attitudes versus the History of Economic Thought, namely preferring Walras over Marshall as the earlier founded "general equilibrium research" or the latter over the French since Marshall created "demand theory."

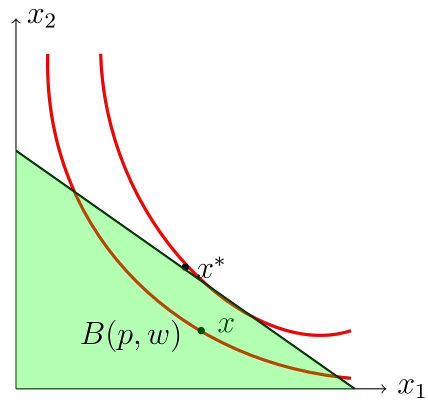


Figure 2.3: The Utility Maximization Problem

3. ***Convexity/Uniqueness:*** *If $\succeq$ is convex, then $u(\cdot)$ is quasi-concave, and $\mathbf{x}(p, w)$ is a convex set. If $\succeq$ is strictly convex, then $u(\cdot)$ is strictly quasi-concave, and $\mathbf{x}(p, w)$ is a singleton.*

Proof. 1) Homogeneity derives directly by the fact that $B(p, w)$ is Homogeneorus of Degree zero. So, if prices and wealth are scaled by a positive coefficient *alpha*, then the budget set does not change. The set of optimal bundles must be the same.

To see 2), notice that $\succeq$ are Locally non-satiated. This means that any open neighborhood of x contains a bundle y such that $y \succ x$. So, if $p \cdot x \leq w$, we have y such that $p \cdot x < p \cdot y < w$, thus contradicting x being the optimal choice.

To see 3), suppose $u(\cdot)$ is quasi-concave, and $x, x' \in x(p, w)$. We want to show that $x'' = \alpha x + (1 - \alpha)x' \in x(p, w)$, for all $\alpha \in [0, 1]$. We know that $\succeq$ is represented by an utility function, so $u(x)$ and since $x, x' \in x(p, w)$, then we can write $u(x) = u(x')$. Further, we know that:

$$p \cdot x'' = p \cdot \left[\alpha x + (1 - \alpha)x'\right] \leq w$$

Since $p \cdot x \leq w$ and $p \cdot x' \leq w$. Therefore, $x'' \in B(p, w)$. Then, since $u(x'') \geq u(x) = u(x')$, $x'' \in x(p, w)$. The argument for strict quasi-concavity is similar. We have seen that $x'' \in B(p, w)$. We want to show that $u(x'') > u(x) = u(x')$. If this is true, then $x, x' \notin B(p, w)$. Therefore, $B(p, w)$ can be at most a singleton. □

It can be extremely useful to assess one important property of demand correspondences (or functions), namely, that they are continuous functions. Then, we have the following result:

Theorem 2.3.1. *If $u(\cdot)$ is C^2 and strictly quasi-concave, then $x(p, w)$ is a continuous function.*

Proof. This is only a sketch of the proof. As seen, if $u(\cdot)$ is quasi-concave and C^2, then $x(p, w)$ is a singleton. If it is a singleton, so it is not empty, then, by Berge's maximum theorem, as long as the objective function is continuous in the variable and in the parameters, and the constraint set is bounded and continuous, then the value function is continuous as well. □

2.3.1 Solving the Consumer's Problem

In this section, I want to provide a general discussion on how to solve the Consumer's problem, also offering some examples.

As seen, a useful assumption on utility functions is that they are continuous and differentiable, other than quasi-concave. Notice that continuity is obtained by continuity of preferences, and if preferences are not continuous, they cannot be represented by utility functions. Differentiability is always assumed because it is useful, but it is not necessary. There can exist utility functions that are not differentiable, but that can generate demand correspondences. If $u(\cdot)$ is C^2, then the Utility Maximization Problem can be solved using the Kuhn-Tucker Method. This is the main method. However, as I will show, in many cases, it is not possible to use the Kuhn-Tucker Method.

Using the Karush-Kuhn-Tucker Method

The Consumer's Problem is a parametric constrained optimization problem, where $\mathbf{x}$ is the variable, and the parameters are given by $\mathbf{p}$ and w. The standard way of solving these types of problems (with inequality constraints) is using the Kuhn-Tucker Method. Then, we set up the so-called Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \lambda) = u(\mathbf{x}) + \lambda\Big(w - \sum_{i=1}^{L} x_i p_i\Big) + \sum_{i=1}^{L} \mu_i x_i$$

Where $\lambda \geq 0$ is called **Lagrange multiplier**. Then, we write down the First-Order Conditions (necessary):

$$\frac{\partial u}{\partial x_i} - \lambda p_i + \mu_i = 0$$

Or equivalently, using the gradient vector of $u(\mathbf{x})$, so:

$$\nabla u(\mathbf{x}) = \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_L} \end{bmatrix} - \lambda \begin{bmatrix} p_1 x_1 \\ \vdots \\ p_L x_L \end{bmatrix} = 0$$

The other necessary conditions are given by **Complementary Slackness**:

$$\lambda\big(w - p_i x_i\big) = 0 \quad \forall i = 1, \ldots, L$$

$$\mu_i x_i = 0$$

When solving the problem, we can have two possible cases: one when all goods x_i are consumed in positive quantities; and a second when (at least) one good x_j is not consumed at all. In the first case, by Complementary Slackness, we have $\mu_i = 0$, and then we can solve the FOCs as follows:

$$\frac{u(\mathbf{x})}{\partial x_i} = \lambda p_i$$

This is equivalent to writing:

$$\frac{\frac{u(\mathbf{x})}{\partial x_i}}{\frac{u(\mathbf{x})}{\partial x_j}} = \frac{p_i}{p_j}$$

And we have **interior solutions**. This expression is the **marginal rate of substitution** of good i for good j, namely the amount of good i that a consumer is willing to give away to buy an additional unit of good j. In the simplified case of $L = 2$, this simply corresponds to the slope of the budget line. This is a condition of optimality because if, for example, the $\mathrm{MRS}_{i,j} > \frac{p_i}{p_j}$, then it could be the case that an increment in consumption of good i could yield a positive change of utility, and then we are not in an optimum.

Example 2.3.1. *An example of a utility function that gives interior optimum is the Cobb-Douglas utility function. Assume for simplicity that there are only two variables x_1, x_2. Then, we have the following problem:*

$$\max_{x_1, x_2 \in X} x_1^{\alpha} x_2^{1-\alpha}$$

$$s.t$$

$$p_1 x_1 + p_2 x_2 \leq w$$

Write the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1^{\alpha} x_2^{1-\alpha} + \lambda \cdot [w - p_1 x_1 + p_2 x_2]$$

By the Kuhn-Tucker Method, we write down the F.O.C.s and the Complementary Slackness Conditions:

- $\frac{\partial \mathcal{L}}{\partial x_i} = 0 \quad \textit{for } i = 1, 2$
- $\lambda \cdot [w - p_1 x_1 - p_2 x_2] = 0$

Write down the F.O.C's:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \alpha \cdot x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 - \mu_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= (1-\alpha) \cdot x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 - \mu_2 = 0 \end{aligned} \tag{2.1}$$

And the Slackness Condition:

$$
\begin{aligned}
&\lambda \cdot \big[w - p_1x_1 - p_2x_2\big] = 0\\
&\mu_1 \cdot x_1 = 0\\
&\mu_2 \cdot x_2 = 0
\end{aligned}
$$

Slackness is satisfied if the multipliers $(\lambda, \mu_1, \mu_2) \geq 0, \gg 0$ *or is equal to* 0. *It is apparent that* μ_1, μ_2 *cannot be different from zero. Indeed, if they are greater than zero, we are in the trivial case where there is no consumption at all. Then, it remains to check for* λ. *If* λ *is equal to zero, the problem simply becomes that of unconstrained optimization of the Lagrangian (the constraint is multiplied by zero, and therefore it disappears). Let's explore the case when* $\lambda \geq 0$. *In this case, the FOCs can be written as:*

$$
\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial x_1} = \alpha \cdot x_1^{\alpha-1}x_2^{1-\alpha} = \lambda p_1\\
&\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) \cdot x_1^{\alpha}x_2^{-\alpha} = \lambda p_2
\end{aligned}
$$

Then, we can solve using the Marginal Rate of Substitution:[4]

$$
\begin{aligned}
&\frac{\alpha x_1^{\alpha-1}x_2^{1-\alpha}}{(1-\alpha)x_1^{\alpha}x_2^{-\alpha}} = \frac{\lambda p_1}{\lambda p_2}\\
&\frac{\alpha}{1-\alpha}x_1^{-1}x_2 = \frac{p_1}{p_2}\\
&\frac{1}{x_1} = \Big(\frac{p_1}{p_2}\Big)\Big(\frac{1-\alpha}{\alpha}\Big)\frac{1}{x_2}\\
&x_1 = \Big(\frac{p_2}{p_1}\Big)\Big(\frac{\alpha}{1-\alpha}\Big)x_2
\end{aligned}
$$

Plug in the budget constraint and solve for x_2*:*

$$
\begin{aligned}
&p_1\Big(\frac{p_2}{p_1}\Big)\Big(\frac{\alpha}{1-\alpha}\Big)x_2 + p_2x_2 = w\\
&\frac{\alpha}{1-\alpha}x_2 + p_2x_2 = w\\
&x_2\Big(\frac{\alpha p_2}{1-\alpha} + p_2\Big) = w\\
&x_2\Big(\frac{p_2}{1-\alpha}\Big) = w\\
&x_2 = \Big(\frac{(1-\alpha)w}{p_2}\Big)
\end{aligned}
$$

[4] Notice that this is not the only method. We can obtain the same result by solving the FOC for λ and then solving the budget constraint. But it is a longer process, and furthermore, provides less economic intuition

Then plug in x_1 and we obtain:

$$x_1 = \Big(\frac{p_2}{p_1}\Big)\Big(\frac{\alpha}{1-\alpha}\Big)\Big(\frac{(1-\alpha)w}{p_2}\Big)$$
$$x_1 = \frac{\alpha w}{p_1}$$

So, then, the Marshallian Demand of the Cobb-Douglas Utility Function is:

$$x(p,w) = \Big(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\Big)$$

This can be generalized, in the case of n-variables, as follows:

$$x(p,w) = \Big(\frac{\alpha_1 w}{p_1}, \dots, \frac{\alpha_n w}{p_n}\Big)$$

Suppose, however, that one good j is not consumed at all. In this case, by complementary slackness, we cannot rule out the possibility that μ_i be greater than zero, so we cannot write $\nabla u(\mathbf{x}) = \lambda \mathbf{p}$, and inequality between the Marginal Rate of Substitution and the price ratio can rise.

Example 2.3.2. *A classical example of when there are corner solutions is given by the so-called perfect substitute goods. Then, we have the following problem:*

$$\max_{x_1, x_2 \in X} x_1 + x_2$$
$$s.t.$$
$$x_1 p_1 + x_2 p_2 \leq w$$

Notice that in this case, we cannot rule out the possibility that x_1 or x_2 be equal to zero. So, setting up the Lagrangian, we have:

$$\mathcal{L}(\mathbf{x}, \lambda) = x_1 + x_2 + \lambda[w - x_1 p_1 - x_2 p_2] + \mu_1 x_1 + \mu_2 x_2$$

Taking the FOCs and CS, we have:

$$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda p_1 + \mu_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_2 + \mu_2 = 0$$
$$\lambda[w - x_1 p_1 - x_2 p_2]$$
$$\mu_1 x_1 = 0$$
$$\mu_2 x_2 = 0$$

Solving the FOCs, we have three cases when $p_1 > p_2$, then all the income is spent in good 1. When $p_1 \leq p_2$, then all the income is spent in good 2. Or finally, if the prices are equal, then the income is distributed between the two goods. Then, the Marshallian demand is:

$$x(p,w) = \begin{cases} \left(\frac{w}{p_1}, 0\right) \\ \left(\frac{w-p_2x_2}{p_1}, \frac{w-p_1x_2}{x_2}\right) \\ \left(0, \frac{w}{p_2}\right) \end{cases}$$

Then we have two corner solutions and one solution that spans the entire budget line.

Finally, I will provide an example of a Quasi-Linear utility function. With this class of utility functions, corner solutions can arise.

Example 2.3.3. *Let's see an example in closed form:*

$$v(p,w) = \max_{x_1,x_2 \geq 0} \ln\left(x_1 + 1 + x_2\right.$$

$$s.t.$$

$$p_1x_1 + x_2 \leq w$$

Notice that the utility function is linear in good 2, and the price of 2 is normalized to 1. Set up the Lagrangian:

$$\mathcal{L}(x_1, x_2, \lambda) = \ln\left(x_1 + 1\right) + x_2 + \lambda\big[w - p_1x_1 - x_2\big]$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1 + 1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1x_1 - x_2 = 0$$

The Complementary Slackness is:

$$\lambda(w - p_1x_1 - x_2) = 0$$

Since we cannot rule out the possibility of corner solutions, we must examine all the possible cases, $x_1, x_2 > 0$, $x_1 > 0, x_2 = 0$ and $x_1 = 0, x_2 > 0$.

Let's see $x_1, x_2 > 0$. From the FOCs,$\lambda = 1$, then:

$$\frac{1}{x_1 + 1} = p_1$$
$$x_1 + 1 = \frac{1}{p_1}$$
$$x_1 = \frac{1}{p_1} - 1$$

Plugging x_1^* into the budget constraint, we have:

$$\begin{aligned}
&p_1\Big(\frac{1}{p_1} - 1\Big) + x_2 = w\\
&1 - p_1 + x_2 = w\\
&x_2 = w - (1 - p_1)
\end{aligned}$$

Then:

$$x_1^* = \frac{1}{p_1} - 1 \quad and \quad x_2^* = w - (1 - p_1)$$

Let's see now $x_1 = 0$ and $x_2 > 0$. The FOCs are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{x_1 + 1} - \lambda p_1 \leq 0\\
\frac{\partial \mathcal{L}}{\partial x_2} &= 1 - \lambda = 0\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w - p_1 x_1 - x_2 = 0
\end{aligned}$$

Notice that the first FOC is less or equal to 0. Then:

$$\begin{aligned}
&1 - p_1 \leq 0\\
&1 \leq p_1
\end{aligned}$$

The demand is:

$$x_1^* = 0 \quad and \quad x_2^* = w$$

Let's see now $x_1 > 0$ and $x_2 = 0$. The FOCs are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{1}{x_1 + 1} - \lambda p_1 = 0\\
\frac{\partial \mathcal{L}}{\partial x_2} &= 1 - \lambda \leq 0\\
\frac{\partial \mathcal{L}}{\partial \lambda} &= w - p_1 x_1 - x_2 = 0
\end{aligned}$$

In this case:

$$x_1^* = \frac{1}{p_2} - 1 \quad and \quad x_2^* = 0$$

From the budget constraint, we have that:

$$p_1 \leq 1 - w$$

Therefore, we can write the Marshallian demand associated to this Quasi-Linear Utility function:

$$\mathbf{x}^*(p, w) = \begin{cases} x_1^* = 0, x_2^* = w \;\; \textit{if } p_1 \geq 1\\ x_1^* = \frac{1}{p_2} - 1, x_2^* = 0 \;\; \textit{if } p_1 \leq 1 - w\\ x_1^* = \frac{1}{p_1} - 1, x_2^* = w - (1 - p_1) \;\; \textit{if } p_1 \in (1 - w, 1) \end{cases}$$

Notice that FOCs and CS are necessary conditions but, in most cases, not sufficient. They became sufficient when $u(\cdot)$ is quasi-concave and monotone, and $\nabla u(\mathbf{x}) \neq 0$ for all $x \in \mathbb{R}^L_+$. To check for Quasi-Concavity, one has to check if the **bordered hessian** matrix is positive.

Solving without the Lagrangian

Sometimes, however, we cannot use the FOCs. The typical scenario when FOCs cannot be used is when there is a non-differentiable utility function, such as a function with some kinks. In that case, other attempts to solve the problem must be guessed, as the following example shows. Unfortunately, there is no general rule to do so. One must mainly rely on economic intuition, trying to guess an interior solution and then find some cases to discuss. In this section, I will provide a general example of complementary goods and a closed-form example.

Example 2.3.4. *We have seen the case of (perfect) complements good as an example of homothetic utility functions. Roughly speaking, this class of utility functions concerns goods that can be consumed only in a linear combination together. The general form this utility function is written is:*

$$u(x_1, \dots, x_n) = \min \big\{x_1, \dots, x_n\big\}$$

Without loss of generality, let's see the case when $n = 2$:

$$u(x_1, x_2) = \min \big\{x_1, x_2\big\}$$

$$s.t.$$

$$p_1 x_1 + p_2 x_2 \leq w$$

Since they are perfect complements, the demand is a kink point, namely $y = x$. Substituting in the budget constraint, we have:

$$\begin{aligned} &p_1 x + p_2 x = w \\ &x(p_1 + p_2) = w \\ &x = \frac{w}{p_1 + p_2} \end{aligned}$$

Therefore:

$$\mathbf{x}^*(p, w) = \frac{w}{p_1 + p_2}, \frac{w}{p_1 + p_2}$$

Example 2.3.5. *Let's see again the case of complements. Let's see now this closed-form example:*

$$\max_{x,y \geq 0} \quad \min \big\{3(x-1), y-2\big\}$$

$$s.t.$$

$$p_x x + p_y y \leq w$$

Since x and y are perfect complements, we can solve the problem as follows:

$$\begin{aligned}
3(x+1) &= y-2 \\
3x+3 &= y-3 \\
y &= 3x-1
\end{aligned}$$

Plugging into the budget constraints, we have:

$$\begin{aligned}
p_x x + p_y(3x-1) &= w \\
p_x x + p_y 3x - p_y &= w \\
x(p_x + p_y 3) &= w + p_y \\
x &= \frac{w+p_y}{p_x+3p_y}
\end{aligned}$$

and y is

$$\begin{aligned}
y = 3 \cdot \left(\frac{w+p_y}{p_x+3p_y}\right) - 1 = \\
\frac{3w+3p_y}{p_x+3p_y} - \frac{p_x+3p_y}{p_x+3p_y} = \\
\frac{3w+3p_y-p_x-3p_y}{p_x+3p_y} = \\
\frac{3w-p_x}{p_x+3p_y}
\end{aligned}$$

Since $x, y \geq 0$, then, to have $y \geq 0$, we must have that $\frac{p_x}{3}$. So, the Marshallian demand associated to this utility function is:

$$\mathbf{x}^*(p,w) = \begin{cases} \frac{w+p_y}{p_x+3p_y}, \frac{3w-p_x}{p_x+3p_y} & \text{if } \frac{p_y}{3} \leq w \\ \frac{w}{p_x}, 0 & \text{if } \frac{p_y}{3} \geq w \end{cases}$$

2.3.2 Comparative Statics and demand correspondence

The main advantage of using the demand function is that this is an object that can be linked to actual choices. This makes it possible to study the effects on demand of the changes in some parameters, like a rise in prices or wealth. For instance, we associate demand theory with the "law of demand": namely, if the price rises, the demand decreases. Mathematically, this means that $\frac{\partial x_i}{\partial p_i} < 0$, $i = 1, \ldots, L$. Using actual demand functions, we can verify this, as well as other hypotheses.

First, however, we need to verify a further property of the demand functions, namely their differentiability.

Theorem 2.3.2. *Suppose that:*

1. $u(\cdot)$ is C^2

2. $u(\cdot)$ is strictly quasi-concave

3. $\nabla u(x) > 0$ for all $x \in \mathbb{R}^L_+$

4. the Bordered Hessian is non-singular, then invertible for all x

Then, the demand function $x(p, w)$ is differentiable.

Proof. (A sketch) A powerful result in the analysis is the **Implicit Function Theorem**.[5] This permits to describe the behavior of an implicit function $f(x, y) = c$, namely a function that cannot be represented as $y = f(x)$ in an open neighborhood (x_0, y_0). This result can be very useful in economics because some functions cannot be represented in closed form. Let's write a utility function $u(\mathbf{x})$ (assume, without loss of generality, that $\mathbf{x} = (x_1, x_2)$. The Lagrangian is $\mathcal{L}(\mathbf{x}, p, w)$. The FOCs can be written as a system of equations depending on (p, w). Then, we can define:

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f_1(\lambda(p, w), \mathbf{x}(p, w), p, w)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = f_2(\lambda(p, w), \mathbf{x}(p, w), p, w)$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = f_3(\lambda(p, w), \mathbf{x}(p, w), p, w)$$

Then, by the Implicit Function Theorem, we can write the $\mathcal{L}(\mathbf{x}, \lambda, p, w)$ as:

$$\mathcal{L}(x(p, w), \lambda(p, w), p, w).$$

Let's apply the theorem:

$$\underbrace{\begin{bmatrix} \frac{\partial \lambda}{\partial w} & \frac{\partial \lambda}{\partial p_1} & \frac{\partial \lambda}{\partial p_2} \\ \frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix}}_{\mathbf{D}_{p,w}(\lambda, \mathbf{x})} = - \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial \lambda} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial \lambda} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial \lambda} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix}^{-1}}_{\mathbf{D}_{\lambda, \mathbf{x}} f_i} \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial w} & \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial p_2} \\ \frac{\partial f_2}{\partial w} & \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial p_2} \\ \frac{\partial f_3}{\partial w} & \frac{\partial f_3}{\partial p_1} & \frac{\partial f_3}{\partial p_2} \end{bmatrix}}_{\mathbf{D}_{p,w} f_i}$$

But notice that $\mathbf{D}_{\lambda, \mathbf{x}} f_i$ can be written as a bordered hessian since the submatrix (2×2) is made up by the second derivatives of the original utility functions. Then, by condition 4) of the theorem, $x(p, w)$ is differentiable. □

Example 2.3.6. *Take a special case of Cobb Douglas, $u(x_1, x_2) = x_1 x_2$. The Consumer Problem is:*

$$\max_{x_1, x_2 \in X} x_1 x_2$$

[5] See the notes on Math for Econ.

$$s.t.$$

$$p_1 x_1 + p_2 x_2 \leq w$$

Take the FOCs:

$$w - p_1 x_1 - p_2 x_2 = 0$$

$$x_2 - \lambda p_1 = 0$$

$$x_1 - \lambda p_2 = 0$$

To know how the demand varies as the parameters vary, we can solve the system without necessarily solving the FOCs and finding the demand. Then, by IFT, we have:

$$\begin{bmatrix} \frac{\partial \lambda}{\partial w} & \frac{\partial \lambda}{\partial p_1} & \frac{\partial \lambda}{\partial p_2} \\ \frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} = - \begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & 0 & 1 \\ -p_2 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -x_1 & -x_2 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix}$$

2.3.3 The Indirect Utility Function

We can define, for each (p, w), the solution to the Utility Maximization Problem as $v(p, w) = u(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the Marshallian demand. The function $v(p, w)$ is called the **indirect utility function**. This is very useful as an analytical tool since it does not depend on preferences over a bundle of alternatives but on actual choices, namely the consumer's demand, which is a function of price and wealth. Therefore, through the indirect utility, we can, for instance, answer policy and welfare questions about actual choices. The following proposition identifies the basic properties of $v(p, w)$.

Proposition 6. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally non-satiated preference relation $\succeq$ on $X = \mathbb{R}^L_+$. Then, $v(p, w)$ is:*

1. *Homogeneous of degree zero*
2. *Strictly increasing in w and non-increasing in p_i for any i*
3. *Quasi-convex (the lower contour set of $v(p, w)$ is convex*
4. *Continuous and differentiable in p and w*

Proof. Proof of continuity and differentiability comes directly from the continuity and differentiability of the demand function. To see that $v(p, w)$ is strictly increasing in w, notice that this, intuitively, means that as long as wealth increases, then indirect utility does. Suppose $\hat{x} \in B(p, w)$, and take $B(p, w + \Delta)$. Then $\hat{x} \in B(p, w + \Delta)$, and then $u(\hat{x} = v(p, w)$. A similar argument goes for non-increasing in p. This means that if p increases, utility does not (but also does not necessarily decrease, it can be constant). $\hat{x} \in B(p, w)$. Then $\hat{x} \in B(p - \Delta), w)$, so $u(\hat{x}) = v(p, w)$. Homogeneity of degree zero is a direct consequence of the same property for $B(p, w)$. Let's see quasi-convexity. Recall the definition:

$$v(\alpha p + (1-\alpha)p', w) \leq \max\Big\{v(p, w), v(p', w)\Big\}$$

In other words, the lower contour set must be convex. This is represented in Figure 4 (notice that this is a price indifference curve, and the consumer gets worse as the price goes up, so as the curve moves toward the right).

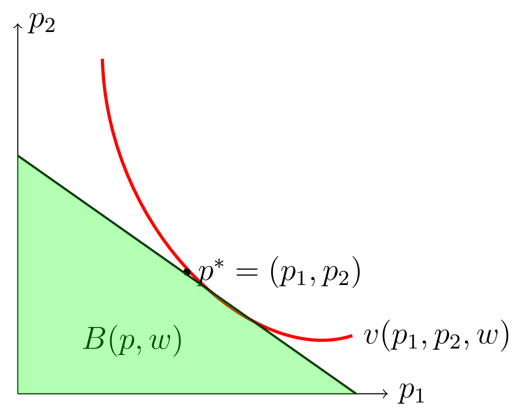


Figure 2.4: A price indifference curve

Take two arbitrary prices p, p' and $\alpha \in [0, 1]$. Then, we can write:

$$\big[\alpha p + (1-\alpha)p'\big]x \leq w$$

$$\alpha px + (1-\alpha)p'x \leq w$$

Then, we can have $p \cdot x \leq w$, $p' \cdot x \leq w$, or both (if both are not true, then this convex combination is not less than w). Then, we can write:

$$x \in B(\alpha p + (1-\alpha)p', w) \subset B(p, w) \cup B(p', w)$$

Notice that $B(p, w) \cup B(p', w)$ is not necessarily a convex set. We can write then:

$$v(\alpha p + (1-\alpha)p', w) = \max_{x \in B(\alpha p + (1-\alpha)p', w)} u(x) \leq \max_{x \in B(p,w) \cup B(p', w)} u(x)$$

We have expanded the feasible set, so in an optimization problem, we are always best off. If $x \in B(p, w)$, then we have $v(p, w)$. if $x \in B(p', w)$, then we have $v(p', w)$. So, we can write:

$$v(\alpha p + (1-\alpha)p', w) \leq \max\Big\{v(p, w), v(p', w)\Big\}$$

$\square$

Example 2.3.7. *Let's see an example of $v(p, w)$ with a (logarithmic) Cobb-Douglas utility function, $u(x_1, x_2) = \alpha \log x_1 + (1 - \alpha) \log x_2$. Recall that:*

$$x_1(p, w) = \frac{\alpha w}{p_1}$$

$$x_2(p, w) = \frac{(1 - \alpha)}{p_2}$$

Then, substituting in $u(x_1, x_2)$, we have:

$$v(p, w) = \alpha \log \frac{\alpha w}{p_1} + (1 - \alpha) \log \frac{(1 - \alpha) w}{p_2}$$

2.4 The Expenditure Minimization Problem

The basic idea of the **expenditure minimization problem** (EMP), (thanks to McKenzie 1957), is that of finding those values of x that minimize the total expenditure, given a certain utility level to be attained. Then, the problem can be formulated as follows:

$$e(p, u) = \min_{x \geq 0} \mathbf{p} \cdot \mathbf{x}$$

$$\text{s.t}$$

$$u(x) \geq 0$$

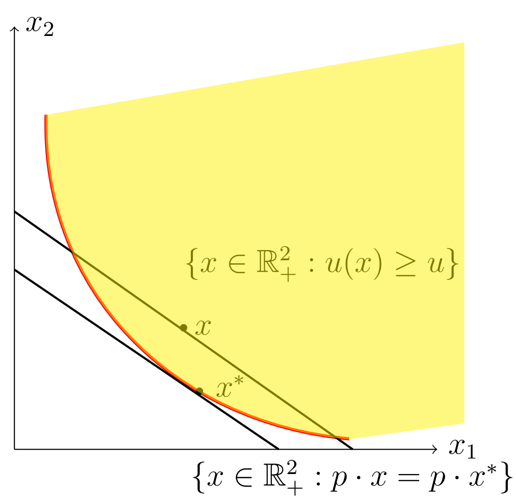


Figure 2.5: The Expenditure Minimization Problem

Whereas the Utility Maximization Problem was about the maximum amount of x needed to maximize utility under a budget constraint, the Expenditure minimization instead is about the minimum amount of expenditure needed to reach a definite level of utility. In other words, to solve the EMP means to seek the minimum amount the consumer must spend at price p to get for himself utility level u. Therefore, the optimal bundle x^* is the bundle that solves the EMP, that is, that minimizes $p \cdot x$ subject to a utility constraint.

Geometrically, it is the point of the set

$$\left\{x \in \mathbb{R}^l_+ : u(x) \geq u\right\}$$

That lies on the least possible budget line associated with a definite price vector (see Figure above).

2.4.1 The Expenditure Function and the Hicksian Demand

The value function for the EMP is given by $e(p, u)$, called **the expenditure function**. Thus $e(p, u)$ gives the minimum expenditure required to achieve utility u at prices p.

The set of optimal consumption bundles in the EMP is known as the **Hicksian Demand Correspondence** (or Function, if univalued), defined as $h(p, u) \in \mathbb{R}^l_+$.

$$h(p, u) = \arg\min_x \sum_{i=1}^{n} p \cdot x \quad s.t. \;\; u(x) \geq u \tag{2.2}$$

In other words, $h(p, u)$ is the set of consumption bundles that the consumer would purchase at prices p if she wished to minimize her expense but still achieve utility u.

Then, exactly like the **Marshallian Demand**, $x(p, w)$ is the solution to the UMP, at given (p, w), $h(p, u)$ is the solution to the expenditure minimization problem at given (p, u).

Let's see now the basic properties of $e(p, u)$ and $h(p, u)$.

Proposition 7. [6] *Suppose $u(.)$ is a continuous utility function representing $\succeq$ Locally Non Satiated and defined on $X = \mathbb{R}^l_+$. Then $e(p, u)$ is:*

1. *Homogeneous of Degree one in p*
2. *Strictly increasing in u and non-decreasing in p_l, $\forall l = 1, \ldots, L$*
3. *Concave in p*
4. *Continuous in p and u*

Proof. To see that $e(p, u)$ is Homogeneous of degree one in p, note that in the EMP if p changes, the utility is unaffected. In other words, the EMP now becomes $\min \alpha \cdot x$ subject to $u(x) \geq u$. If x^*, then $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p, u)$.

[6] Proposition 3.E.2, Mas-Colell, Whinston, and Green 1995, pp. 59–60

$e(p,u)$ being not strictly increasing in u means that if u increases, then the value of $e(p,u)$ does not. To see this, assume x' and x'' as optimal consumption bundles for the utility levels $u(x')$ and $u(x'')$, where $u(x'') > u(x')$ and $p \cdot x'' < p \cdot x'$. Take a bundle $\hat{x} = \alpha x''$ (with $\alpha \in (0,1)$). By continuity of $u(.)$, if $\alpha \sim 1$, then $u(\hat{x}) > u(x')$ and $p \cdot \hat{x} < p \cdot x'$. But then, x' is not optimal in the EMP.

Let's see now that $e(p,u)$ is not decreasing in princes. This means that when p decreases, $e(p,u)$ does not. Assume p'' and p', where $p''_l \geq p'_l$ and $p''_k = p'_k \forall l \neq k$. Let x'' be an optimizing consumption bundle in the EMP for prices p''. Then $e(p'',u) = p'' \cdot x'' \geq p' \cdot x' = e(p',u)$

To see concavity, assume $\bar{u}$ and $p'' = \alpha p + (1-\alpha)p'$, (with $\alpha \in [0,1]$). Suppose x^* is optimal in the EMP at prices p''. Then:

$$\begin{aligned}
&e(p'',\bar{u}) = p'' \cdot x^* = \\
&\big[\alpha p + (1-\alpha)p'\big]x^* = \\
&\alpha p \cdot x^* + (1-\alpha)p' \cdot x^* \geq \\
&\geq \alpha e(p,\bar{u}) + (1-\alpha)e(p',\bar{u})
\end{aligned} \tag{2.3}$$

Whereas: $\alpha p \cdot x^* + (1-\alpha)p'x^* \geq \alpha e(p,\bar{u}) + (1-\alpha)e(p',\bar{u})$, $\alpha e(p,u^*) * (1-\alpha)e(p',u^*)$, and $u^* = u(x^*) > \bar{u}$ □

The Hicksian Demand instead has three basic properties:

Proposition 8. [7] *Suppose $u(\cdot)$ is a continuous utility function representing $\succeq$ Locally Non Satiated and defined on $X = \mathbb{R}^l_+$. Then, for any $p >> 0$ $h(p,u)$ is:*

1. *Homogeneous of Degree Zero in p*
2. *No excess utility: $\forall x \in h(p,u), u(x) = u$*
3. *Convexity/uniqueness: if $\succeq$ is convex, then $h(p,u)$ is a convex set; and if $\succeq$ is strictly convex so that $u(.)$ is strictly quasi-concave, then there is a unique element in $h(p,u)$.*

Proof. Since the constraint is the same in the EMP for $(\alpha p, x)$ and (p,x), then:

$$\min_{u(x) \geq u} \alpha p \cdot x = \alpha \min_{u(x) \geq u} p \cdot x$$

Suppose that there is some $x \in h(p,u)$ such that $u(x) > u \geq u(0)$. Take a bundle $x' = \alpha x$, with $(\alpha \in (0,1)$. Then $p \cdot x' < p \cdot x$, and since $u(\cdot)$ is continuous (by the Intermediate Value Theorem), there is an α such that $u(x') \geq u$. This contradicts the assumption that $x \in h(p,u)$.

Note that $h(p,u) = \{x \in \mathbb{R}^l_+ : u(x) \geq u\} \cup \{x : p \cdot x = e(p,u)\}$ is the intersection to two convex sets, and hence is convex. If preferences are strictly convex, $x, x' \in h(p,u)$, then for $\alpha \in [0,1], x'' = \alpha x + (1-\alpha)x' \succ x$ and $p \cdot x'' = e(p,u)$. But this contradicts "no excess utility." □

[7] Proposition 3.E.3, Mas-Colell, Whinston, and Green 1995, pp. 59–60

Intuitively, the meaning of concavity is simply that if there is an optimal consumption bundle in the EMP, whose value is given by $e(p, u)$, if p changes so that the new price vector is p', then the upper bound of the new consumption bundle is given by $p' \cdot x$, a linear transformation of $p \cdot x$.

Another result allows us to link $h(p, u)$ and the **Compensated Law of the Demand**. In a nutshell, demand and prices move in opposite directions for price changes that are accompanied by **Hicksian Wealth Compensations**. This means that $h_k(p, u)$ is decreasing in p_k, i.e., Hicksian Demand is always downward sloping. Note that this is not always true in the case of the Marshallian, where, for example, we can find such situations as those involving Giffen Goods (the demand rises as the price rises).

The following proposition links the Hicksian Demand and the compensated law of demand.

Proposition 9. [8] *Suppose $u(\cdot)$ is a continuous utility function representing $\succeq$ Locally Non Satiated and defined on $X = \mathbb{R}^l_+$, and $h(p, u)$ is uni-valued, for any $p >> 0$. Then $h(p, u)$ satisfies the Compensated Law of Demand: for all p and p':*

$$(p' - p) \cdot [h(p', u) - h(p, u)] \leq 0 \tag{2.4}$$

Proof. In the EMP, at prices p, $h(p, u)$ is optimal. This means that it allows the consumer to attain the same level of utility, but with a lesser expenditure. That is:

$$\begin{aligned} p' \cdot h(p', u) \leq p' \cdot h(p, u) \\ p \cdot h(p, u) \leq p \cdot h(p,' u) \end{aligned} \tag{2.5}$$

Subtracting these equations yields the equation (4). Indeed:

$$\begin{aligned} &p' \cdot h(p', u) - p' \cdot h(p, u) - p \cdot h(p, u) + p \cdot h(p,' u) = \\ &(p' - p) \cdot [h(p', u) - h(p, u)] \leq 0 \end{aligned} \tag{2.6}$$

□

We can see why the **Hicksian Demand** is always **downward sloping** in the Figure below. The original prices for x_1 and x_2 determine a bundle set $B(p, w)$ and $h(p, u)$ is h^A. As prices change to reach the same utility, the new bundle set becomes $B(p', w)$. So then, the new $h(p, u)$ is h^B, which is still on the indifference Curve I. This is because the Hicksian Demand refers to an EMP problem, so then, given a utility level, the rational consumer must find the best way of reaching it.

In the Marshallian demand setting, instead, the problem is different since the constraint is the budget set. If prices change, and so does the budget set, the new optimal x^* lies on a different indifference curve.

[8] Proposition 3.E.4. Mas-Colell, Whinston, and Green 1995, pp. 62–3

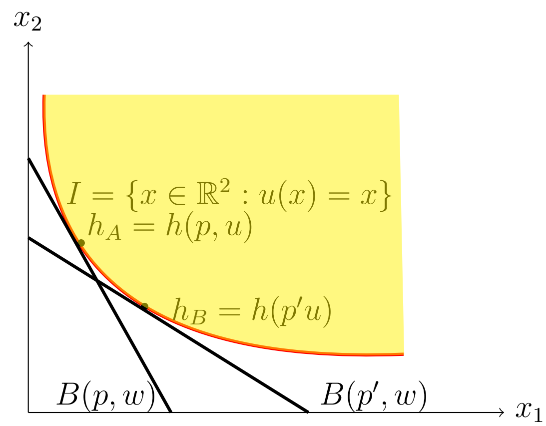


Figure 2.6: Changes in the Hicksian Demand as prices change

Example 2.4.1. *Let's see an example. Take a log Cobb-Douglas Utility Function. Then, the Expenditure Minimization Problem is the following:*

$$e(p, u) = \mathbf{p} \cdot \mathbf{x}$$

$$s.t$$

$$\alpha \ln(x_1) + (1 - \alpha) \ln(x_2) \geq u$$

Where $\alpha \in (0, 1)$, $p_1, p_2 > 0$ *and* $u \in \mathbb{R}$. *Set up the Lagrangian:*

$$\mathcal{L}(x_1, x_2, \lambda) = p_1 \cdot x_1 + p_2 \cdot x_2 + \lambda \Big[u - \alpha \ln(x_1) - (1 - \alpha) \ln(x_2) \Big] + \mu_1 x_1 + \mu_2 x_2$$

Since it is a Cobb-Douglas Utility function, we can exclude corner solutions. Then, writing down the FOCs and Complementary Slackness:

$$\frac{\partial \mathcal{L}}{\partial p_1} = p_1 - \lambda \frac{\alpha}{x_1} = 0 \Rightarrow \frac{x_1}{\alpha} \lambda = \frac{1}{p_1} = x_1 = \frac{\alpha}{p_1} \lambda$$

$$\frac{\partial \mathcal{L}}{\partial p_2} = p_2 - \lambda \frac{1 - \alpha}{x_2} = 0 \Rightarrow \frac{x_2}{1 - \alpha} \lambda = \frac{1}{p_2} = x_2 = \frac{1 - \alpha}{p_2} \lambda$$

Plugging x_1 *and* x_2 *in the budget constraint, we have:*

$$\alpha \ln\Big(\frac{\lambda \alpha}{p_1}\Big) + (1 - \alpha) \ln\Big(\frac{(1 - \alpha)\lambda}{p_2}\Big) = u$$

$$\Big(\frac{\lambda \alpha}{p_1}\Big)^{\alpha} \Big(\frac{1 - \alpha)\lambda}{p_2}\Big)^{1-\alpha} = e^u$$

Since λ^{α} *and* $\lambda^{1-\alpha}$, *we cannot take them out from the brackets, and furthermore, they sum to 1. So:*

$$\lambda \Big(\frac{\alpha}{p_1}\Big)^{\alpha} \Big(\frac{1 - \alpha)}{p_2}\Big)^{1-\alpha} = e^u$$

$$\lambda = \Big(\frac{p_2}{\alpha}\Big)^{\alpha}\Big(\frac{p_2}{1-\alpha)}\Big)^{1-\alpha} e^u$$

Plugging λ in x_1 and x_2, and solving, gives us:

$$\begin{aligned} x_1 =& \Big(\frac{p_1}{\alpha}\Big)^{\alpha}\Big(\frac{p_2}{1-\alpha}\Big)^{1-\alpha} \cdot \frac{1}{p_1} \cdot \alpha \cdot e^u \\ & p_1^{\alpha-1}\Big(\frac{p_2}{1-\alpha}\Big)^{1-\alpha} e^u \alpha^{1-\alpha} \\ & \Big(\frac{\alpha p_2}{p_1(1-\alpha)}\Big)^{1-\alpha} e^u = h_1(p,u) \end{aligned} \tag{2.7}$$

and, following the same logic:

$$\begin{aligned} x_2 =& \Big(\frac{p_1}{\alpha}\Big)^{\alpha}\Big(\frac{p_2}{1-\alpha}\Big)^{1-\alpha} \cdot \frac{1-\alpha}{p_2} \cdot e^u \\ & \Big(\frac{p_1}{\alpha}\Big)^{\alpha}(1-\alpha)^{\alpha} p_2^{-\alpha} e^u \\ & \Big(\frac{(1-\alpha)p_1}{p_2\alpha}\Big)^{\alpha} e^u = h_2(p,u) \end{aligned} \tag{2.8}$$

Therefore, we can write down the expenditure function:

$$\begin{aligned} e(p,u) =& \mathbf{p} \cdot h(p,u) = p_1 \cdot h_1(p,u) + p_2 \cdot h_2(p,u) \\ & p_1\Big(\frac{\alpha p_2}{p_1(1-\alpha)}\Big)^{1-\alpha} e^u + p_2\Big(\frac{(1-\alpha)p_1}{p_2\alpha}\Big)^{\alpha} e^u = \\ & e^u\Big[p_1^{\alpha}\Big(\frac{\alpha p_2}{1-\alpha}\Big)^{1-\alpha} + p_2^{1-\alpha}\Big(\frac{1-\alpha)p_1}{\alpha}\Big)^{\alpha}\Big] = \\ & e^u p_1^{\alpha} p_2^{1-\alpha}\Big[\Big(\frac{\alpha}{1-\alpha}\Big)^{1-\alpha} + \Big(\frac{1-\alpha}{\alpha}\Big)^{\alpha}\Big] = \\ & e^u p_1^{\alpha} p_2^{1-\alpha}\Big[\frac{\alpha^{1-\alpha}\cdot\alpha^{\alpha} + (1-\alpha)^{\alpha}\cdot(1-\alpha)^{1-\alpha}}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}\Big] = \\ & e^u p_1^{\alpha} p_2^{1-\alpha}\Big[\frac{\alpha + (1-\alpha)}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}\Big] = \\ & e^u p_1^{\alpha} p_2^{1-\alpha}\Big[\frac{1}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}\Big] = \\ & e^u p_1^{\alpha} p_2^{1-\alpha}\Big[(1-\alpha)^{\alpha-1}\alpha^{-\alpha}\Big] \end{aligned} \tag{2.9}$$

It is easy to see that strictly increases in u and p_i. To see that it is Homogeneous of degree 1 in p:

$$e(\alpha p, u) = \alpha p_1 \cdot h_1(p,u) + \alpha p_2 h_2(p,u) =$$

$$\alpha p_1\Big(\frac{\alpha p_2}{p_1(1-\alpha)}\Big)^{1-\alpha} e^u + \alpha p_2\Big(\frac{(1-\alpha)p_1}{p_2\alpha}\Big)^{\alpha} e^u =$$

$$e^u\Big[\alpha p_1^\alpha\Big(\frac{\alpha p_2}{1-\alpha}\Big)^{1-\alpha}+\alpha p_2^{1-\alpha}\Big(\frac{1-\alpha)p_1}{\alpha}\Big)^\alpha\Big]=$$
$$e^u\alpha p_1^\alpha \alpha p_2^{1-\alpha}\Big[\Big(\frac{\alpha}{1-\alpha}\Big)^{1-\alpha}+\Big(\frac{1-\alpha}{\alpha}\Big)^\alpha\Big]=$$
$$e^u\alpha\big(p_1^\alpha p_2^{1-\alpha}\big)\Big[\frac{\alpha^{1-\alpha}\cdot\alpha^\alpha+(1-\alpha)^\alpha\cdot(1-\alpha)^{1-\alpha}}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\Big]=$$
$$\alpha e^u\big(p_1^\alpha p_2^{1-\alpha}\big)\Big[\frac{\alpha+(1-\alpha)}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\Big]=$$
$$\alpha e^u\big(p_1^\alpha p_2^{1-\alpha}\big)\Big[\frac{1}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\Big]=$$
$$\alpha e^u\big(p_1^\alpha p_2^{1-\alpha}\big)\Big[(1-\alpha)^{\alpha-1}\alpha^{-\alpha}\Big]=\alpha e(p,u)$$

2.4.2 The dual problem

The Expenditure minimization problem (EMP) is the dual of the Utility maximization problem (UMP). Then, we have the following important result:

Proposition 10. [9] *Let $u(\cdot)$ be a continuous utility function representing $\succeq$ Locally Non Satiated and defined on $X=\mathbb{R}^l_+$, and that price vector is $p>>0$. Then:*

1. *If x^* is optimal in the UMP when $w>0$, then x^* is optimal too in the EMP, when the required utility level is $u(x^*)$, and the minimized expenditure level in this EMP is w*
2. *If x^* is optimal in the EMP when the required utility level is $u>u(0)$, then x^* is optimal in the UMP when wealth is $p\cdot x^*$. Moreover, the maximized utility level in this UMP is u*

Proof. Let's start with 1) We can show this by contradiction. Assume x^* is not optimal in the EMP. Then, it exists a x' such that $u(x')>u(x^*)$, and $p\cdot x<p\cdot x^*\leq w$. Still, however, $\succeq$ are LNS, so that we can find an x'' such that $u(x'')>u(x')$ and $p\cdot x'<w$. But then $x''\in B(p,w)$ and $u(x'')>u(x^*)$. This contradicts the assumption of x^* being optimal in the UMP. Finally, since x^* solves the UMP when prices are p, then $p\cdot x^*=w$.

Let's see 2) Since $u>u(0)$, then $x^*>0$ and $p\cdot x^*>0$. Suppose x^* is not optimal. Then there exist a $x'>x^*$ such that $u(x')>u(x^*)$ and $p\cdot x'<p\cdot x*$. Take a bundle $x''=\alpha x'$ (with $\alpha\in(0,1)$). By continuity of $u(.)$, if $\alpha\sim 1$, then $u(x")>u(x^*)$ and $p\cdot x"<p\cdot x^*$. But this contradicts the optimality of x^* in the EMP. Then x^* must be optimal in the UMP when $w=p\cdot x$ and $\max_x u=u(x^*)$. □

[9] Proposition 3.E.3, Mas-Colell, Whinston, and Green 1995, pp. 59–60

Note finally that a solution to the EMP always exists under very general conditions: the constraint set must be non-empty.

From the duality property, we can recover the following results. Fixing (p, w), for all $p << 0$ and $u \geq u(0)$, by definition, $v(p, u) = \max_{x \in B(p,w)} u(x)$, and therefore $v(p, w) = u(x^*)$. That is, $v(p, w)$ is the maximum utility attainable at price p and wealth w. $e(p, u)$ then can be thought of as the minimum expenditure needed to attain utility $v(p, u)$. So we can write:

$$e(p, v(p, u)) \leq w$$

Furthermore, by Walras' Law $p \cdot x^* = w$. So, we can have $e(p, v(p, u)) = w$. Similarly, as $e(p, u)$ is the minimum spending to attain utility u, we can write $v(p, e(p, u) \geq u$, since v is the highest utility achievable given income $e(p, u)$.

From the results above, we can relate the Hicksian Demand and the Marshallian Demand as follows:

- $h(p, u) \equiv x(p, e(p, u))$, since $e(p, u) \equiv e(p, v(p, w)) \equiv w$. The Marshallian Demand at income w is equal to the Hicksian Demand at utility $v(p, w)$.
- $x(p, w) \equiv h(p, v(p, w))$ since $v(p, w) \equiv v(p, e(p, u)) \equiv u$. The Hicksian Demand at utility u is the same as the Marshallian Demand at income $e(p, u)$.

In particular, the last identity is important since it shows that the Hicksian Demand is equal to the Marshallian Demand at the minimum income necessary, at the given prices, to achieve the desired level of utility. Therefore, the Hicksian Demand is simply the Walrasian Demand function for the various goods if the consumer's income is "compensated" so as to achieve some target level of utility.

These identities further imply that for a fixed price vector p, $e(p, \cdot)$ and $v(p, \cdot)$ are one the inverse of the other. This means that:

$$e(p, u) = v^{-1}(p; u)$$

Since $v(p, w)$ strictly increases in wealth, and $v(p, e(p, u)) = u$. And:

$$v(p, w) = e^{-1}(p; w)$$

Since $e(p, u)$ strictly increases in u, and $e(p, u(p, w)) = w$.

Finally, there are two results that make it possible to recover the Hicksian demand from the expenditure function and the Marshallian demand from the indirect utility. Both these results exploit the **Envelope Theorem**[10]and the properties of continuity and differentiability of $v(p, w)$ and $e(p, u)$.

[10]Roughly speaking, this theorem states that if we can express the optimal solution of the consumer problem as a differentiable parametric function, then if we change some parameters of the objective, changes in the optimizer do not contribute to the change in the objective function. In the case of the consumer's problem, since $v(p, u) = \max_{x \in B(p,w)} u(x)$, then, we can write:

$$\frac{dv(p)}{dp_i} = \frac{\partial \mathcal{L}(x^*; p)}{\partial p}$$

More in the Math notes.

The first of these two results is the **Shephard's Lemma**.

Proposition 11 (Shephard's Lemma)**.** *Suppose that $u(\cdot)$ is a continuous utility function representing Locally Non Satiated preference $\succeq$ and suppose that $h(p,u)$ is a function. Then, the $e(p,u)$ is differentiable in p, and for all $i = 1\ldots,n$*

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u) \tag{2.10}$$

Proof. As seen above, $e(p,u)$ is the value function associated to the EMP. Therefore, we can write:

$$e(p,u) = \min_{x \geq 0} \mathbf{p} \cdot \mathbf{x}$$

$$\text{s.t}$$

$$u(x) \geq u$$

Taking the Lagrangian of the equation above gives:

$$\mathcal{L}(p,u,\lambda) = \sum_{j=1}^{n} p_j \cdot x_j + \lambda[u - u(x)]$$

Let's apply now the Envelope Theorem:

$$\frac{\partial e(p,u)}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = x_i^* \quad \forall x^* \in h(p,u) \tag{2.11}$$

Where x_i^* is the Hicksian Demand for good x_i □

A result similar to Shephard's Lemma, but for Utility Maximization and Marshallian Demand, is the **Roy's Identity**. This presents a way to derive the Marshallian Demand Function of good i for some consumer from the Indirect Utility Function, $v(p,w)$ of that consumer.

Proposition 12 (Roy's Identity)**.** *Let $u(\cdot)$ be continuous and representing LNS and strictly convex $\succeq$, and $u(\cdot)$ is differentiable. Then:*

$$x_i(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} \quad \text{for } i = 1, \ldots, k$$

Proof. [11] We know that if x^* is optimal in the UMP, then it is optimal also in the EMP. Therefore we can write :

$$x(p,w) \equiv h(p,u)$$

[11] Based on the proof in Varian 1992, pp. 106–7

at given p, w, u. Furthermore, we know also that:

$$u \equiv v(p, e(p, u))$$

That is, no matter what the prices are, if the consumer has the minimal income to get utility u, at prices p, then the maximal utility is u.

We can differentiate with respect to p and obtain:

$$\frac{\partial v(p, e(p, u))}{\partial p_i} + \frac{\partial v(p, e(p, u)}{\partial p_i} \cdot \frac{\partial e(p, u)}{\partial p_i} = 0$$

Note that:

$$\frac{\partial v(p, e(p, u)}{\partial p_i} \cdot \frac{\partial e(p, u)}{\partial p_i} \equiv \frac{\partial v(p, w)}{\partial w} \cdot \frac{\partial e(p, u)}{\partial p_i}$$

and

$$\frac{\partial e(p, u)}{\partial p_i} \equiv h_i(p, u) \equiv x_i(p, w)$$

These identities hold for all p, w. Therefore, by rearranging, we have:

$$-\frac{\partial v(p, w)}{\partial w} x_i(p, w) = \frac{\partial v(p, w)}{\partial p_i} =$$

$$x_i(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}}$$

□

Example 2.4.2. *Let's see again the example of the log Cobb-Douglas utility function:*

$$u(x_1, x_2) = \alpha \ln(x_1) + (1 - \alpha) \ln(x_2)$$

Recall that:

$$x_1 = \frac{\alpha w}{p_1}$$

$$x_2 = \frac{(1 - \alpha)}{p_2}$$

Then, $v(p, w)$ is:

$$v(p, w) = \alpha \ln\left(\frac{\alpha w}{p_1}\right) + (1 - \alpha) \ln\left(\frac{(1 - \alpha)w}{p_2}\right) =$$

$$v(p, w) = \alpha \ln\left(\frac{\alpha}{p_1}\right) + (1 - \alpha) \ln\left(\frac{(1 - \alpha)}{p_2}\right) + \ln(w)$$

Since $e(p, u) = w$ and $v(p, w) = u$, and rearranging terms, we can write:

$$\ln e(p, u) = u - \alpha \ln\left(\frac{\alpha}{p_1}\right) - (1 - \alpha) \ln\left(\frac{(1 - \alpha)}{p_2}\right)$$

Solving for $e(p,u)$:

$$e(p,u) = \frac{e^u}{\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{(1-\alpha)}{p_2}\right)^{1-\alpha}}$$
$$e(p,u) = e^u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{(1-\alpha)}\right)^{1-\alpha}$$
$$e(p,u) = e^u p_1^{\alpha} p_2^{\alpha}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right]$$

Notice that this is the same result of (9)*. To find the Hicksian demand, we can use the identity $h(p,u) = x(p, e(p,u))$. Then:*

$$\begin{aligned} x_1(p, e(p,u)) =& \frac{\alpha}{p_1} e(p,u) = \\ & \frac{\alpha}{p_1} e^u p_1^{\alpha} p_2^{\alpha}\left[\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\right] = \\ & e^u \frac{p_2^{1-\alpha}}{p_1^{1-\alpha}}\left[(1-\alpha)^{\alpha-1}\alpha^{1-\alpha}\right] = \\ & e^u \frac{p_2^{1-\alpha}}{p_1^{1-\alpha}}\left[\left(\frac{\alpha}{(1-\alpha)}\right)^{1-\alpha}\right] = \\ & e^u\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} = h_1(p,u) \end{aligned} \tag{2.12}$$

This is equal to (10)*. Following the same logic, we can find $h_2(p,u)$.*

Finally, notice that using Roy's Identity, we obtain $x_1(p,w)$ and $x_2(p,w)$. Indeed:

$$\begin{aligned} x_1(p,w) = & -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = \\ & -\frac{-\frac{\alpha}{p_1}}{\frac{1}{w}} = \frac{\alpha w}{p_1} \end{aligned} \tag{2.13}$$

2.5 The Slutsky Equation

At this point one could question what is the meaning of the results above, and more in general, of all the Consumer Theory in such a mathematical fashion. The object of consumer theory must be to analyze how a rational consumer reacts when he faces some changes in prices and wealth. There are some results (listed below) that describe the rational consumer's behavior with regard to her Marshallian Demand. However, in order to fully assess this point, it is not sufficient to rest upon the UMP. Indeed, the total change can be decomposed into two parts, one that involves the Marshallian Demand and one that involves the Hicksian Demand.

Note, however, that the Hicksian Demand is not directly observable (one of its parameters is u). Still $h(p,u)$ is computable through the Marshallian Demand, which is observable (in principle). We have seen that there are some important results in recovering Hicksian Demand and Marshallian Demand from the Expenditure Function and Indirect Utility (i.e., Shephard's Lemma and Roy's identity). It is important now to relate $h(p, u)$ and $x(p, w)$ in a more general way in order to make possible a detailed analysis of how a change in the prices affects the change in the demand.

It is easy to have some intuition on why changes in prices or wealth have some effect on the demand. For what concerns the Marshallian Demand, i.e., the solution to the UMP for all prices and income levels, there are some important results worth briefly recapping.

The **Wealth Effects** indicates how the demand changes when wealth changes. This is represented by the following $(1 \times L)$ vector:

$$D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in \mathbb{R}^L \tag{2.14}$$

The **Price Effects** instead shows how the change in the price of one good affects the demand for all the goods. The following square matrix represents these effects:

$$D_p x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} \tag{2.15}$$

Finally, these **Substitution Effects** (i.e., Wealth Effects and Price Effects) can be expressed by the following square matrix called **Slutsky Matrix**:

$$S(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p, w) & \dots & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} x_L(p, w) \\ \vdots & \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} x_1(p, w) & \dots & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} x_L(p, w) \end{bmatrix} \tag{2.16}$$

The results above make clear a very simple and intuitive fact. Assume a change in prices, say a raising of p_k. Then the consumer faces two different situations: first, the good k is more expensive relative to other goods, so one can expect a decline in k's consumption, and depending on the relation between k and other goods, it could be the case that even their consumption falls. In any case, there is a "substitution" or "cross-substitution" effect. Second, the consumer's real income has declined. If k is more expensive, the more it raises, the less can be spent on the other goods of the bundle. The issue is how to explore this result analytically.

One way of thinking about this problem is how to **compensate** the consumer for the increase of p_k by giving her some Δw so that her real income is the same as before. This allows us to isolate the effect of a shift in relative prices from the effect a change

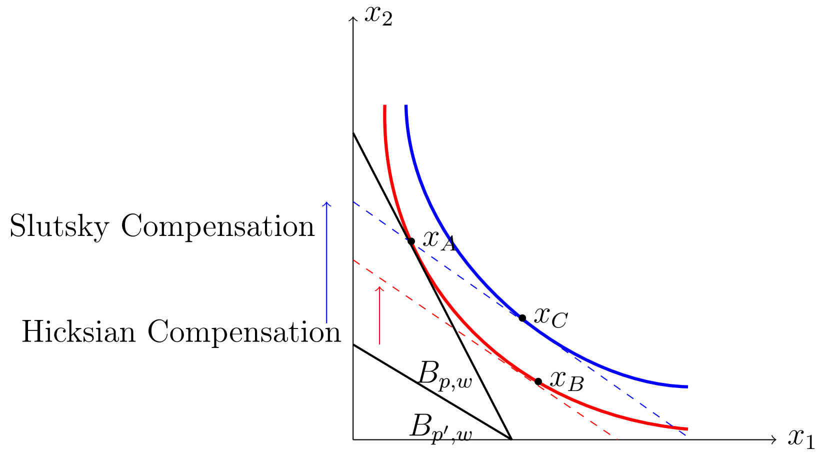


Figure 2.7: Slutsky and Hicksian Compensation

in prices has on the real income. The problem is now of determining how much Δw must be. There are two ways of answering this question:

- **Slutsky Compensation** is that Δw_S such that the consumer can buy back her original old optimal bundle x. This can be written as $w + \Delta w_s$. Note, however, that the consumer can now choose a different bundle since the old one is no longer optimal.

- **Hicksian Compensation** is that change in wealth, Δw_h, which allows the consumer to maintain her utility. Still note that, as seen before, the Hicksian Demand satisfies the Compensated Law of Demand, and therefore, as apparent in Figure 2, Hicksian Demand is a form of compensated demand.

These compensations are represented graphically in Figure 7. The consumer's original demand, at (p, w) is x_A. Then, the price of x_2 rises so that the new budget set is $B(p', w)$. To compensate the consumer in order to stay on the original indifference curve, her new real income must be added Δw_h (Hicks Compensation) so to reach the red dashed line (the Hicksian Compensation Budget Line). The new demand is x_B. In order instead of making the old demand x_A allowable, the consumer must be compensated with Δw_s (Slutsky Compensation) to reach the dashed blue line (the Slutsky Compensation Budget Line). Still note that x_A is no longer an optimal bundle so the new demand is x_C.

This graph makes it apparent that a change in the price of x_2 affects the demand of x_1 in a way that involves both the Hicksian Compensation and Slutsky. From the result above we can write the fundamental **Slutsky Equation**.

Proposition 13 (The Slutsky Equation)**.** [12] *Suppose that $u(.)$ is a continuous utility function representing a L.N.S. and Strictly Convex $\succeq$ defined on $X = R^L_+$. Then, for all (p, w) and $u = v(p, w)$ we have:*

[12] Proposition 3.G.3, Mas-Colell, Whinston, and Green 1995, pp. 71–2)

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \tag{2.17}$$

Proof. Recall that: $w \equiv e(p, v(p,w))$ and $h(p,u) \equiv x(p.e(p,u))$. Differentiating with respect to p_k we have:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,e(p,u)}{\partial p_k} + \frac{\partial x_l(p,e(p,u))}{\partial p_k}\frac{\partial e(p,u)}{\partial p_k}$$

Still, $\frac{\partial x_l(p,e(p,u))}{\partial p_k} \equiv \frac{\partial x_l(p,w)}{\partial w}$ and, by Shephard's Lemma, $\frac{\partial e(p,u)}{\partial p_k} \equiv h_k(p,u)$ and, finally $h_k(p,u) \equiv x_k(p,w)$.

Then we have the result:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

□

The importance of the Slutsky Equation is that it decomposes the demand change induced by a price change into two separate effects: the **Substitution Effect** and the **Income Effect**.

$$\underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution Effect}} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{\text{Total Effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Income Effect}}$$

Furthermore (17) can be arranged in a more economically meaningful way as follows:

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution Effect}} - \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Income Effect}}$$

The economic intuition behind the Slutsky Equation is that if the price of good k increases, this has two effects on the demand for good l: the **Substitution Effect**, a movement along the original indifference curve since the utility is fixed (i.e., Hicksian Demand refers to an EMP). And an **Income Effect**, that is, the movement from one indifference curve to another. A change in prices determines a change in income and therefore in the size of the budget line, which represents the constraint of the UMP.

These effects are represented graphically in Figure 8. A consumer faces an initial price-wealth situation (p,w) and therefore a budget set $B(p,w)$.Then he chooses x_A. Let's assume now a change in the price of x_1, so then the new budget set is $B(p',w)$. The new optimal consumption is x_C. But this move from x_A to x_C can be decomposed into two different parts. The **Substitution Effect**, which affects the Hicksian Demand. Since, by definition, $h(p,u)$ solves the EMP constrained to $u(x)$, the new demand must

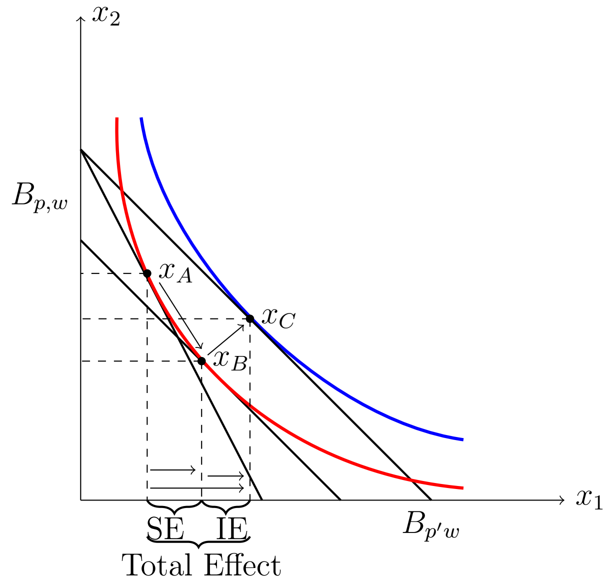


Figure 2.8: Total Effect, Substitution Effect, and Income Effect

be on the same indifference curve. But since the lower price of x_1 makes it possible to reach a higher indifference curve, that is, the new demand is x_C.

The importance of the Slutsky Equation is that it allows us to view comparative statics in prices as the sum of an income effect and a substitution effect. However, in order to fully assess this point, one has to look at these effects in a more general way, that is, by rewriting equation (17) in matrix form. Indeed note that its right part is an element of the so-called **Slutsky Matrix**.

$$
\begin{gathered}
D_p h(p,u) = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \dots & \frac{\partial h_1(p,u)}{\partial p_L} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_L(p,u)}{\partial p_1} & \dots & \frac{\partial h_L(p,u)}{\partial p_L} \end{bmatrix} = \\
\begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) & \dots & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} x_L(p,w) \\ \vdots & \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} x_1(p,w) & \dots & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \end{bmatrix} = S(p,w)
\end{gathered} \tag{2.18}
$$

The advantage of this matrix form is that it shows the own-price substitution effects as well as the cross-price substitution effects. $D_p h(p,u)$ is the matrix of price effects for the Hicksian Demand, which is, roughly speaking, the equivalent of the Hicksian Demand of the matrix of the Price Effects for the Marshallian Demand. However, there are also important differences. Since we know that we can recover $h_i(p,u)$ simply by differentiating the expenditure function $e(p,u)$, this means that if we are at an optimum in the EMP, the changes in demand caused by price changes do not affect the consumer's expenditure.

Example 2.5.1. *Let's continue with a log Cobb-Douglas Utility function,* $u(x_1, x_2) = \alpha \ln(x_1) + (1-\alpha)\ln(x_2)$*. Recall that the Mashallian demand functions are:*

$$x_1 = \frac{\alpha w}{p_1}$$

$$x_2 = \frac{(1-\alpha)w}{p_2}$$

and the Hicksian demand for good 1 *is:*

$$h_1(p,u) = e^u\Big(\frac{\alpha p_2}{(1-\alpha)p_1}\Big)^{1-\alpha}$$

Let's see the impact of the change of price of good 1 *on* x_1*. We can compute:*

$$\begin{aligned}
&\frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) = \\
&\underbrace{-\frac{\alpha w}{p_1^2}}_{\text{Total effect}} + \underbrace{\frac{\alpha}{p_1}\cdot\frac{\alpha w}{p_1}}_{\text{Income effect}} = \\
&-\frac{aw + a^1 w}{p_1^2} = \underbrace{\alpha w \frac{(\alpha - 1)}{p_1^2}}_{\text{Substitution effect}}
\end{aligned} \tag{2.19}$$

Let's see the effect on the Hicksian demand of good 1*. We can rewrite* $h_1(p,u)$ *as:*

$$h_1(p,u) = (\alpha p_2)^{1-\alpha} \cdot ((1-\alpha)p_1)^{\alpha - 1}$$

Then:

$$\begin{aligned}
\frac{\partial h_1(p,u)}{\partial p_1} =&(\alpha-1)(\alpha p_2)^{1-\alpha}((1-\alpha)p_1)^{\alpha-2}(1-\alpha) = \\
&\frac{(\alpha-1)(\alpha p_2)^{1-\alpha}(1-\alpha)^{\alpha-2+1}}{p_1^{2-\alpha}} = \\
&(\alpha-1)\frac{(\alpha p_2)^{1-\alpha}(1-\alpha)^{\alpha-1}}{p_1^{2-\alpha}} = \\
&(\alpha-1)\frac{(\alpha p_2)^{1-\alpha}}{(1-\alpha)^{1-\alpha}p_1^{2-\alpha}}
\end{aligned}$$

Notice that $p_1^{2-\alpha} = p_2 \cdot p_1^{1-\alpha}$*. Then:*

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{(\alpha-1)}{p_1}\Big[\frac{\alpha p_2}{(1-\alpha)p_1}\Big]^{1-\alpha} e^u$$

Recall that:

$$e^u = w\Big(\frac{\alpha}{p_1}\Big)^{\alpha}\Big(\frac{1-\alpha)}{p_2}\Big)^{1-\alpha}$$

Therefore, plugging e^u *in the expression gives:*

$$\frac{(1-\alpha)}{p_1}\Big[\frac{\alpha p_2}{(1-\alpha)p_1}\Big]^{1-\alpha}\cdot w\Big(\frac{\alpha}{p_1}\Big)^{\alpha}\Big(\frac{1-\alpha)}{p_2}\Big)^{1-\alpha}$$

Solving it gives:

$$\begin{aligned}
&\frac{(\alpha-1)}{p_1}\Big[\frac{\alpha p_2}{(1-\alpha)p_1}\Big]^{1-\alpha}\cdot w\Big(\frac{\alpha}{p_1}\Big)^{\alpha}\Big(\frac{1-\alpha)}{p_2}\Big)^{1-\alpha}=\\
&\frac{\alpha-1}{p_1}\cdot\frac{(\alpha p_2)^{1-\alpha}}{(1-\alpha)^{1-\alpha}p_1^{1-\alpha}}\cdot\frac{\alpha^{\alpha}}{p_1^{\alpha}}\cdot\frac{(1-\alpha)^{1-\alpha}}{p_2^{1-\alpha}}\cdot w=\\
&\frac{(\alpha-1)}{p_1}\cdot\frac{\alpha^{1-\alpha}}{p_2^{1-\alpha-(1-\alpha)}}\cdot\frac{\alpha^{\alpha}}{p_1^{\alpha}}\cdot(1-\alpha)^{1-\alpha-(1-\alpha)}w=\\
&\frac{(\alpha-1)\cdot\alpha^{1-\alpha}\cdot\alpha^{\alpha}w}{p_1^{1+(1-\alpha)+\alpha}}=\\
&\frac{(\alpha-1)\alpha w}{p_1^2}
\end{aligned}$$

Which is the result previously obtained in (19).

Let's see the effect of the change of the price of good 1 *on* x_2. *Recall that* x_2^* *is:*

$$x_2 = \frac{(1-\alpha)w}{p_1}$$

Then $\frac{\partial x_2(p,w)}{\partial p_1} = 0$. *The income effect is:*

$$\frac{\partial x_2(p,w)}{w}\cdot x_1(p,w) = \frac{1-\alpha}{p_2}\frac{\alpha w}{p_1}$$

The Slutsky equation is:

$$\frac{\partial h_2(p,w)}{\partial p_1} = 0 - \frac{1-\alpha}{p_2}\frac{\alpha w}{p_1}$$

In other words, since the optimal demand x_i^* *associated with a Cobb-Douglas utility function depends only on the price* p_i, *a raise of* p_{-i} *does not have an effect on the demand for good* i. *The substitution effect (i.e., the change in the Hicksian demand) and the income effect for the Marshallian demand are equal.*

The Slutsky Matrix associated to this utility function is:

$$S(p,w) = \begin{bmatrix} \frac{\alpha w(\alpha-1)}{p_1^2} & 0-\frac{1-\alpha}{p_2}\frac{\alpha w}{p_1} \\ 0-\frac{1-\alpha}{p_2}\frac{\alpha w}{p_1} & \frac{(1-\alpha)w(1+(1-\alpha))}{p_2^2}\end{bmatrix}$$

The Slutsky matrix is important because it is directly computable from the Marshallian demands that are (generally) observable since they are functions of observable variables, prices, and wealth. Furthermore, if $h(p,u)$ is continuously differentiable at (p,u), we can state this important result concerning the properties of $D_p h(p,u)$.

Proposition 14. *Suppose that $u(.)$ is a continuous utility function representing L.N.S. and strictly convex preferecnces relation $\succeq$ defined on $X = \mathbb{R}^L_+$. Suppose also that $h(p,u)$ is continuously differentiable at (p,u). Then:*

1. *$D_p h(p,u) = D_p^2 e(p,u)$*

2. *$D_p h(p,u)$ is a Negative Semidefinite Matrix*

3. *$D_p h(p,u)$ is a symmetric matrix*

4. *$D_p h(p,u)p = 0$*

Proof. This is a scratch of the proof. 1) follows from Shephard's Lemma by differentiation. 2) This property, as well as 3, derives from the fact that since $e(p,u)$ is continuous and concave, its Hessian Matrix is symmetric (by the properties of the Hessian Matrices) and Negative Semidefinite (since the function is concave). 4) This derives from the fact that $h(p,u)$ is Homogeneous of Degree Zero in p. Then $h(\alpha p,u) = h(p,u)$, and therefore $h(\alpha p,u) - h(p,u) = 0$. Taking the derivative with respect to α we have $\frac{\partial h(\alpha p,u)}{\partial \alpha} p = 0$.

□

The economic meaning of Negative Semidefiniteness of $D_p h(p,u)$ is that if the price of i rises, then the change in $h_i(p,u)$ is not positive (i.e., $\frac{\partial h_i(p,u)}{\partial p_i} \leq 0$). This is a differential form of the law of the demand.[13] The symmetry of $D_p h(p,u)$ is a direct consequence of its being a Hessian Matrix, but its economic meaning is somewhat "fuzzy." Indeed, it means that the effect of a small increase in the price of good i on the quantity demanded of good j is identical to the effect of a small change in the price of j on the quantity demanded of i.

Example 2.5.2. *Let's take the Slutsky matrix from the Log Cobb-Douglas Utility:*

$$S(p,w) = \begin{bmatrix} \frac{\alpha w(\alpha-1)}{p_1^2} & 0 - \frac{1-\alpha}{p_2}\frac{\alpha w}{p_1} \\ 0 - \frac{1-\alpha}{p_2}\frac{\alpha w}{p_1} & \frac{-(1-\alpha)w(1-(1-\alpha))}{p_2^2} \end{bmatrix}$$

It is apparent that this matrix is symmetric. To check for Negative semidefiniteness is less trivial. One way (among the others) is to check for all principal minors, namely each submatrix obtained eliminating $(n-k)$ rows and $(n-k)$ columns. Then, if at least one minor on the main diagonal has a determinant equal to 0, and the other minors'

[13]Because in the Hessian Matrix in the main diagonal, we find all the second-order derivatives, and in the Negative Semidefinite Matrix these elements are always non-positive

determinants, $(-1)^k|S|^k$, are ≤ 0 if k is odd, and ≥ 0 if k is even, the matrix is negative semi-definite. In this example:

$$\Big|\frac{\alpha w(\alpha - 1}{p_1^2}\Big|, \Big|0 - \frac{1-\alpha}{p_2}\frac{\alpha w}{p_1}\Big| \leq 0$$

and:

$$\begin{aligned}|S(p,w)| = & \frac{-\alpha w + \alpha^2 w}{p_1^2} \cdot \Big[\frac{-(1-\alpha)w[1-(1-\alpha)]}{p_2^2}\Big] - \Big(\frac{(1-\alpha)}{p_2}\frac{\alpha w}{p_1}\Big)^2 = \\ & \big[(\alpha w)^2 - 2(\alpha w)(\alpha^2 w) + \alpha^2 w\big] - \big[(\alpha w)^2 - 2(\alpha w)(\alpha^2 w) + \alpha^2 w\big] = 0\end{aligned}$$

Then the matrix is negative semidefinite.

Example 2.5.3. *An interesting case for comparative statics is that of* ***Giffen goods*** *(Kreps 1990, p. 61). Recall that goods are classified with respect to wealth and price effects. Then, good j:*

- *It is* ***inferior*** *if its wealth-derivative is less than zero: $\frac{\partial x_j}{\partial w} < 0$*
- *It is* ***normal*** *if its wealth-derivative is greater and equal than zero: $\frac{\partial x_j}{\partial w} \geq 0$.*
- *It is* ***ordinary*** *if its price-derivative is less than zero: $\frac{\partial x_j}{\partial p_j} < 0$.*
- *It is* ***Giffen*** *if its price-derivative is greater than zero: $\frac{\partial x_j}{\partial p_j} > 0$.*

Let's consider the case of a Giffen good. Since good is Giffen with respect to its own price change, we have to look for those values across the main diagonal in all the matrices involved. So, in Slutsky Equation's terms:

$$\frac{\partial x_j(p,w)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_j} - \frac{\partial x_j}{\partial w} \cdot x_j(p,w)$$

$\frac{\partial x_j}{\partial p_j}$ must be greater than zero. But we know that $\frac{\partial h_j}{\partial p_j}$ is non-positive since it is an element of the main diagonal of a Hessian Negative Semi-Definite Matrix. Therefore, the only possibility for j's own price-derivative of being > 0 is that j is also inferior, i.e., that $\frac{\partial x_j}{\partial w} < 0$. But this is not sufficient. j must be sufficiently inferior so that its income effect overcomes the substitution effect. This could be the case if j occupies a great share in the consumer's consumption bundle.

However, notice that although a theoretical possibility, there is neither historical nor empirical evidence of the existence of such Giffen goods.

From the Slutsky matrix, it is easy to identify different cross-price effects. Indeed it is sufficient to look at the sign of cross-derivatives. Then two goods, l, k are **substitutes** at (p,u) if $\frac{\partial h_l}{\partial p_k} \geq 0$; and **complementary** if $\frac{\partial h_l}{\partial p_k} \leq 0$. Finally, since $D_p h(p,u)$ is Negative

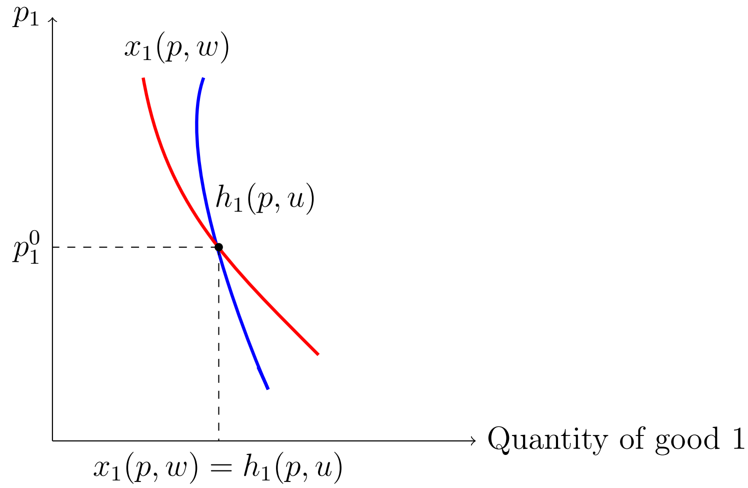


Figure 2.9: The Marshallian and the Hicksian Demand for a Normal Good

Definite, and therefore $\frac{\partial h_i}{\partial p_i} \leq 0$, property 4 of Proposition 10 ensures that there must be a good k for which $\frac{\partial h_l(u,p)}{\partial p_k} \geq 0$, i.e. every good has at least one substitute.

There is another interpretation of the Slutsky Equation. Indeed it describes the relationship between the slope of the Hicksian demand curve and the Marshallian demand curve at prices p. This relationship is represented in Figure 9, for the case of a Normal Good. This represents the demand curve for the good 1, holding all other prices fixed. Note that the two demands are equal when p_1 is p_1^0. Furthermore, in the figure, Marshallian demand and Hicksian demand refer to the same utility level, i.e., $h_1(p, v(p,w)) = x_1(p,w))$. From the figure, it is apparent that the slope of the Walrasian demand curve is less negative than the slope of the Hicksian demand for that price. That is the Hicksian demand curve is less responsive to price changes than is the Marshallian demand curve. At level p_1^0 there is no income compensation. When $p > p_1^0$, income compensation is positive because the individual needs help to remain at the same utility level. Finally, at $p < p_0^1$, the income compensation is negative to prevent an increase in utility from a lower price.

To understand this, let's see the Slutsky equation again. Recall that the own-price derivative is negative by definition for the Hicksian demand. In order for the Marshallian demand to have a lesser slope than the Hicksian, the income effect must be positive. Therefore, a good 1 must be normal. In the case of inferior goods, the relationship is reversed: the Hicksian demand is less negatively steeper than the Marshallian demand.

2.6 Welfare Evaluation of price changes

Economists want to measure how consumers are affected by changes in prices and wealth. So far we have seen how consumers react to these changes. The issue is now to provide a way of measuring it. The simplest way of addressing this problem is by the notion of **Consumer Surplus** (see below). However, this measure is mainly imprecise, and only in specific circumstances (addressed below) can it be considered exact.

The first problem to deal with is that we cannot really measure how utility changes as an effect of some policy. To simplify, we consider a consumer with rational, continuous, and Locally Non-Satiated $\succeq$, and furthermore, that both $e(p, u)$ and $v(p, w)$ are differentiable. Besides, the only focus will be on a price change so that the wealth is fixed, and it is evaluated the impact of a welfare change from p^0 to p^1.

Let's take (p^0, w) and (p^1, w), that is the pair representing the original prices and wealth and the pair representing new prices and the same wealth. A simple way of seeing it is to compute the variation in the consumer's indirect utility:

$$v(p^1, w) - v(p^0, w)$$

Above there is, intuitively, the welfare change. If the difference between utility at new prices and old wealth and old prices and old wealth is positive, then we could presume the consumer has benefitted from this change.

However, we don't know what utility is, and the way we constructed utility functions aimed only to make ordinal utility representable. A possible getaway from this point is that of linking utility to money, using what is usually referred to as **Money Metric Indirect Utility**, which is constructed starting from the $e(p, u)$, and has the same properties (see Proposition above).

Thus, choosing a price vector $\bar{p} >> 0$, and an indirect utility function $v(p, w)$, we can write the following:

$$e(\bar{p}, v(p, w))$$

Therefore, we can write the utility difference as follows:

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$$

This function gives how much money is needed, at prices $\bar{p}$ to reach utility $v(p, w)$. In other words, this function measures how much income the consumer would need, at prices $\bar{p}$ to be as well off as she would be facing prices p and income w.

Assume that we are facing a change of prices from p^0 to p^1 (so that $\bar{p}$ can be either the new prices or the old ones). Then, the question is: what is the impact on a given consumer, with an income w, of the change of p^0 to p^1?

Two measures of compensation can be employed. These are the **Compensating Variation** and the **Equivalent Variation**. We can define both of them in terms of $e(p, u)$ and $v(p, w)$. Recall that $v(p, w) \equiv u$.

So then, we can write CV as follows:

$$CV(p^0, p^1, w) = e(p^0, u^0) - e(p^1, u^0) = w - e(p^1, u^0) \tag{2.20}$$

Since $e(p^0, u^0) = e(p^1, u^1) = w$ (and $v(p^1, w) = u^1$ and $v(p^2, w)$, if the prices change from p^0 to p^1, CV tells how much we will have to compensate, or charge, the consumer to stay on the same indifference curve. It uses the new prices as the base.

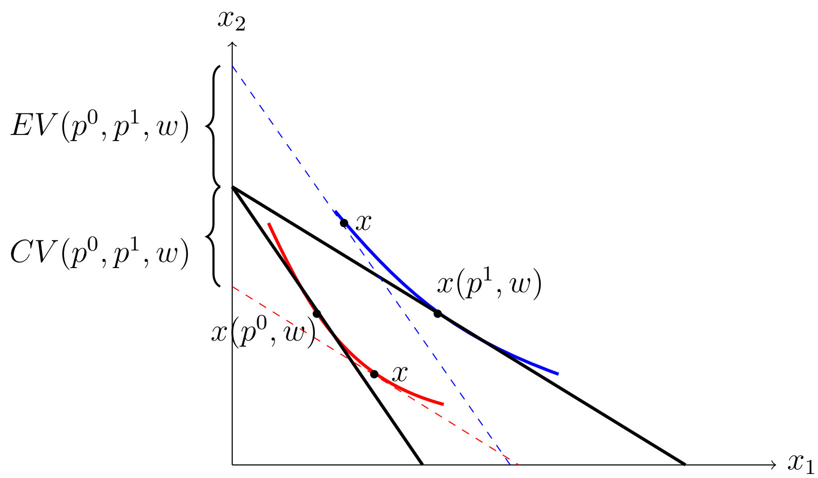


Figure 2.10: The Equivalent Variation and the Compensating Variation

Equivalently, EV can be written as:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w \tag{2.21}$$

That is, the change in expenditure that would be required at the original prices to have the same effect on consumers that price change had. In other words, it uses the current prices as the base and asks what income change at the current prices would be equivalent to the proposed change in terms of its impact on utility. An equivalent definition, (Kreps 2013, p. 296) is the following:

Definition 2.6.1. The Equivalent Variation (EV) is the amount of money that must be added to w so that the consumer is indifferent between having w at prices p^1 and having $w + EV$ at prices p^0.

The Compensating Variation (CV) is the amount of money that must be subtracted from w so that the consumer is indifferent between having w at prices p^0 and having $w - CV$ at prices p^1.

These variations are depicted in Figure 10. Each indifferent curve is associated with a level of utility. Therefore, each budget set represents those combinations (p, w) through which the consumer obtains utility u and u^1. Assume that the price of x_1 decreases from p^0 to p^1. Now the consumer can reach a new indifference curve, therefore he can obtain higher utility. EV represents how much the consumer must be compensated in order to be as well off as when facing p^1. CV instead represents how much money should be taken away from the consumer in order to make her stay as well off as when facing p^0.

Recall that the classic tool for measuring welfare changes is **Consumer's surplus**:

$$CS = \int_{p^0}^{p^1} x(t)dt$$

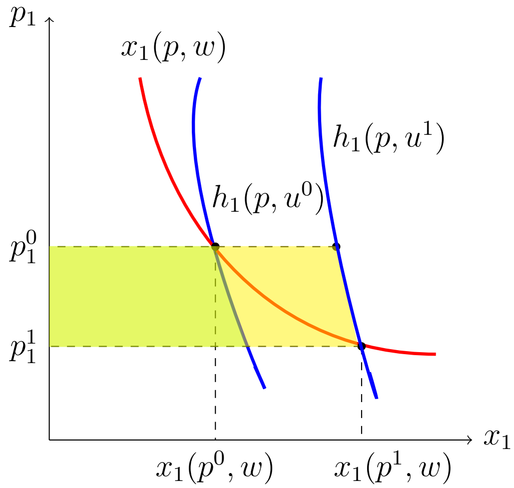


Figure 2.11: The Equivalent Variation and the Compensating Variation for a Normal Good

Assuming a demand function $x(p)$, the Consumer's surplus associated with a price movement from p^0 to p^1 is the area to the left of the demand curve between p^0 and p^1. However, with one exception (if preferences are quasi-linear), usually, CS is not a precise measure of welfare changes because $EV \neq CV$.

The EV and CV can be represented in terms of the Hicksian Demand Curve. Assuming that only the price of good 1 changes from p^0 to p^1, and $w = w^1 = w^0$, we can write:

$$EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_y(p_1, \bar{p}_{-1}, u^1) dp_1 \tag{2.22}$$

Where $\bar{p}_{-1} = (p_2, \ldots, p_L)$. Thus the change in Consumer Welfare measured by the equivalent variation is represented by the area between p^0, p^1 and the left of the Hicksian Demand for good 1 associated with utility level u^1, that is the green and yellow area in Figure 11.

Similarly, the Compensating Variation can be written as:

$$CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_y(p_1, \bar{p}_{-1}, u^0) dp_1 \tag{2.23}$$

This is the area between p^0, p^1 and the Hicksian Demand for good 1 at utility u^0, which is the green area in Figure 11.

In other words, we can say that the Compensating Variation is the integral of the Hicksian Demand curve associated with the initial level of utility, and the Equivalent Variation is the integral of the Hicksian demand curve associated with the final level of utility.

Assuming, as done in Figure 11, that good 1 is normal, $EV > CV$. This relation reverses in the case of good 1 being inferior.

Furthermore, if preferences are quasi-linear (i.e., there is no wealth effect for good 1), $CV = EV$. In this last case (and only in this one), $CV = EV$ corresponds to the Consumer Surplus. In all other cases, this can be seen as no more than an approximation between Compensating Variation and Equivalent Variation.

Chapter 3

General Equilibrium in a Pure Exchange Economy

3.1 Introduction

The idea of general economic equilibrium dates back to the origins of political economy, to the idea that different individuals, in pursuing their own self-interest, would benefit the society as a whole. This idea is embodied in the famous example of the Scottish philosopher and one of the founders of Political Economy, Adam Smith, namely the notorious "invisible hand."(Smith 1776) Put in a modern fashion, the problem is how a vast number of individuals aggregate in such a way that efficient allocation of resources and productive efforts arise without, apparently, any authority coordinating their decisions. Then, it emerged the idea that **prices** played a role in that process since all the individuals face the same price and have different responses towards them: for some, the price is too high, for some too low, some they can afford, some cannot, and so on.

The idea of the working of prices in a single market, and therefore of **equilibrium** in it, has been treated and partially formalized in the second half of the XIXth century, giving rise to the popular idea of (partial) equilibrium as the intersection between supply and demand. (Marshall 1920) This idea, however, refers to a single market. In the same period, several attempts were made to extend this framework to study the working and interplay of **all the markets** in an economy since, intuitively, what happens in the market for, say, cars, is influenced by what happens in the market for fuel, labor, houses, and so on...

The first work to address these issues was due to Leon Walras (Walras 2014), and despite his way of addressing it largely lacked mathematical rigor, since then, the problem is defined as **Walrasian Equilibrium** (other than competitive equilibrium. In these notes the terms will be interchangeable).

The idea of General Economic Equilibrium raised two different questions: first if this equilibrium was possible (e.g., if it existed); second, how it was characterized, namely what are its properties in terms of efficiency, especially when compared to allegedly alternative systems, like planned economies.

For a relatively long period, after Walras' first address to this issue, the idea of existence was deemed "trivial", and the focus was on the efficiency properties of the economic equilibrium.[1] However, rigorous attempts to establish the mathematical foundations of Economics showed that this was not true (Wald 1951; Neumann 1945), and a rigorous proof of the existence of General Economic Equilibrium was reached only in 1954, after some important results concerning its properties, the so-called **Welfare Theorems** have already been proved. (Arrow and Debreu 1954; McKenzie 1954)

Therefore, in the following notes, I will start by providing a series of definitions characterizing both the very abstract set-up that we are using, that of **pure-exchange economy**, and the Walrasian Equilibrium. Later, I will discuss the Welfare theorem and finally the main result concerning General Economic Equilibrium, namely the mathematical proof of its existence.

3.2 Pure Exchange Economy: the Walrasian Model

The pure-exchange economy is the simplest and most abstract setting that we can use to fully characterize an economic system where there are a finite (but potentially a very large) number of agents, each with endowments and preferences that can be represented by utility functions. The main idea is that we can solve a general equilibrium problem using the **Walrasian equilibrium** (or Competitive equilibrium), a solution concept that, in a nutshell, requires that all agents be price-takers and utility maximizers. Having characterized an economy in terms of equilibria, we can infer some ideas concerning its optimal properties.

3.2.1 The primitives of the model

Let's consider an economy with I agents $i \in \mathcal{I} = 1, \dots, I$, and commodities $l = 1, \dots, L \in \mathcal{L}$. We define a bundle of commodities as a vector $x \in \mathbb{R}^L_+$. Each agent has an endowment $e^i \in \mathbb{R}^L_+$, and an utility function $u : \mathbb{R}^m_+ \to \mathbb{R}$. Therefore, an economy can be fully characterized as:

$$\mathcal{E} = (u_i, e^i)_{i \in \mathcal{I}}$$

Agents are price-takers, and $p \in \mathbb{R}^L_+$ are non-negative. Each agent is utility maximize; namely, she solves the following problem:

$$\max_{x \in \mathbb{R}^L_+} u^i(x)$$

s.t.

[1] In a nutshell, Walras' idea was that of building a model where the number of equations was equal to the number of variables. This would have automatically provided the existence of a solution. Of course, this is not true so since the 1920s, some attempts to introduce further restrictions, like the impossibilities of negative prices, were attempted, for instance by Gustav Cassel (Cassell 1932). But these were still not sufficient.

$$p \cdot x \leq p \cdot e^i$$

Where $p \cdot e^i$ represents her endowment at current prices. Therefore, the budget set is:

$$B^i(p, e^i) = \left\{ x : p \cdot x \leq p \cdot e^i \right\}$$

In the analysis of demand theory, we have established several results linking preferences to utility functions and the solution to a Utility Maximization Problem (or an Expenditure Minimization) to the existence of a demand function (or correspondence). In that setup, there was only one consumer facing different prices or price changes. In general equilibrium, there are potentially many consumers, still the analysis does not change too much. Each consumer has a utility function and solves a constrained optimization problem. However, for the sake of simplicity, some assumptions are made regarding the shape and properties of utility functions, as well as the endowments.

Definition 3.2.1. The following are the main assumptions about consumer preferences and endowments:

A.1 For all agents $i \in \mathcal{I}, u^i(x)$ is continuous

A.2 For all agents $i \in \mathcal{I}$, $u^i(x)$ is increasing, namely $u^i(x) > u^i(x')$ if $x \gg x'$

A.3 For all agents $i \in \mathcal{I}$, $u^i(x)$ is concave

A.4 For all agents $i \in \mathcal{I}, e^i \gg 0$

The last assumption says that every agent has at least a very small amount of endowments, namely there is no agent in the economy whose $e^i = (0, \ldots, 0)$. A further implicit assumption is the lack of externality, namely the utility of each agent is affected only by her consumption, and not others.

Notice that $A1$ and $A4$ are technical assumptions in order to make things simpler. $A2 - A3$ can be weakened, as we will see later. In particular, $A2$ makes it possible to use Walras' Law, and $A3$ to make aggregate demand-correspondence convex-valued.

3.2.2 The Walrasian Equilibrium

A **Walrasian equilibrium** for an economy is a vector of prices and quantities (consumption bundles) (p^*, x^*) such that each agent maximizes her utility function, and each market clears. In particular, the last assumption means that, for each good, the total demand among all agents is equal to the total supply. Thus, we have the following formal definition.

Definition 3.2.2. A Walrasian equilibrium for the economy $\mathcal{E} = (u_i, e^i)_{i \in \mathcal{I}}$ is a vector $\{p, \{x^i\}_{i \in \mathcal{I}}\}$, such that:

- each agent i maximizes her utility given p:

$$x^i \in \arg\max_{x \in B^i} u^i(x)$$

- each good's market $l \in \mathcal{L}$ clears:

$$\sum_{i \in \mathcal{I}} x_l^i = \sum_{i \in \mathcal{I}} e_l^i$$

Notice that in this very simple and very abstract setting, each agent is a price-taker, so the impact of her choice on prices and other agents' demands is risible. Prices form instantaneously as the effect of the matching between supply and demand in each market. To represent the idea of price formation, Walras introduces the fictitious figure of the auctioneer and the process of *tatonnement*, namely at the beginning of each time period, prices were formed as the outcome of some auction process, and therefore taken as given by other agents. In the most abstract analysis of General Equilibrium, we can avoid model price formation. However, an important feature concerning equilibrium prices must be noted.

Corollary 3.2.0.1. *If (p^*, x^*) is a Walrasian equilibrium, then $(\lambda p^*, x^*)$ with $\lambda > 0$, is an equilibrium too.*

This corollary is important because it allows for price normalization, and this will be helpful in the proof of existence. The most common normalizations are $\lambda = \frac{1}{p_i}$ such that the equilibrium price system becomes $p^* = (1, p_2, \ldots, p_L)$; and $\lambda = \frac{1}{||p||}$, so that we can write p^* as belonging to the simplex formed by $L-1$ prices, namely Δ^{L-1}.

Example 3.2.1. *Let's see a simple example. There are two people, Ann and David, and two goods, chips and beer each with some endowment and with some preferences over it. To make things simpler, they have the same utility functions and differ only in endowments. Then, Ann's utility is given by the following familiar Cobb-Douglas Utility function:*

$$u_A(x_B^A, x_C^A) = \alpha \ln(x_C^A) + (1-\alpha)\ln(x_B^A)$$

And David's preferences are represented by:

$$u_D(x_B^D, x_C^D) = \beta \ln(x_C^D) + (1-\beta)\ln(x_B^D)$$

Where $\alpha, \beta \in (0,1)$. Ann's endowment is $e^A(1,2)$ (one can of beer and two packs of chips), and David's is $e^D = (2,1)$. Each agent's problem is the following:

$$\max_{x_B, x_C \in \mathbb{R}_+^2} \alpha \ln x_C + (1-\alpha)\ln x_B$$

$$s.t$$

$$p_B x_B + p_C x_C \leq p_B(1) + p_C(2)$$

Therefore Ann's demand (recall that is Cobb-Douglas utility function) is:

$$x_B^A = \frac{\alpha(p_B + 2p_C)}{p_B} \quad and \quad x_C^A = \frac{(1-\alpha)(p_B + 2p_C)}{p_C}$$

Similarly, for David:

$$x_B^D = \frac{\beta(2p_B + p_C)}{p_B} \quad \textit{and} \quad x_C^D = \frac{(1-\beta)(2p_B + p_C)}{p_C}$$

In equilibrium markets clear for each good. So for beer, we have:

$$x_B^A + x_C^D = e_B^A + e_B^D$$

that is:

$$\frac{\alpha(p_B + 2p_C)}{p_B} \frac{\beta(2p_B + p_C)}{p_B} = 3$$

So the equilibrium price is the ratio:

$$\frac{p_C^*}{p_B^*} = \frac{3 - \alpha - 2\beta}{2\alpha + \beta}$$

Notice that one market clearing condition solves the price for the second market. Solving for the prices and plugging in the demand functions, we have the equilibrium allocations.

3.2.3 A graphical example for a 2-goods and 2-agents economy: the Edgeworth Box

A useful way to graphically study General Equilibrium, but only for the case of 2 goods and 2 agents is using the so-called **Edgeworth boxes**.

These are represented in Figure 1. There are two agents A and B, and each has an endowment $e^A = (e_x^A, e_y^A)$ and $e^B = (e_x^B, e_y^B)$ (represented by q). Any point in the box represents nonwasteful allocations of x, y for the two agents. The line passing by q is the equilibrium price and the budget line for the two agents. This line divides the box into two different budget sets for each agent. As seen in consumer theory, the Marshallian demand is the point of tangency between the indifference curve and the budget line for the two agents. Therefore the point where the two demands match is the efficient allocation of this economy.

Edgeworth boxes can be used to describe several economies. See for example the following situation, where there is a unique equilibrium. Suppose two agents have the following utility functions: $u^A = x_1$ and $u^B = x_2$, and the following endowments, $e^A = (0, 1)$ and $e^B = (1, 0)$. Namely, agent 1 receives utility only from the consumption of x_2 but has only a unit of x_1. The reverse for agent 2. This situation can be represented in Figure 2a.

The initial endowment is e. However, A increases her utility by pushing her indifference curve toward the right. B increases her utility by pushing her indifference curve toward the top (notice that for B the plot is mirrored). This process of "adjustment" will stop with the feasible set, namely at the competitive equilibrium in the figure. In this equilibrium, $p_A = p_B$.

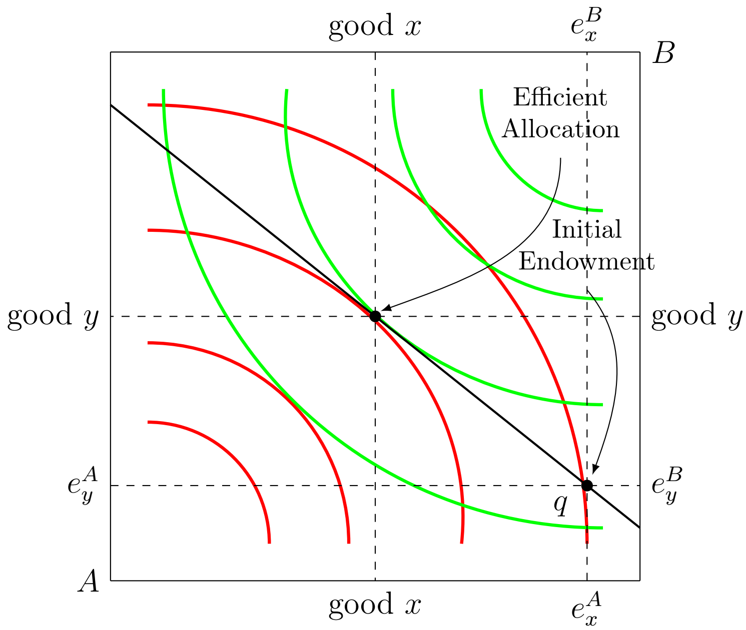


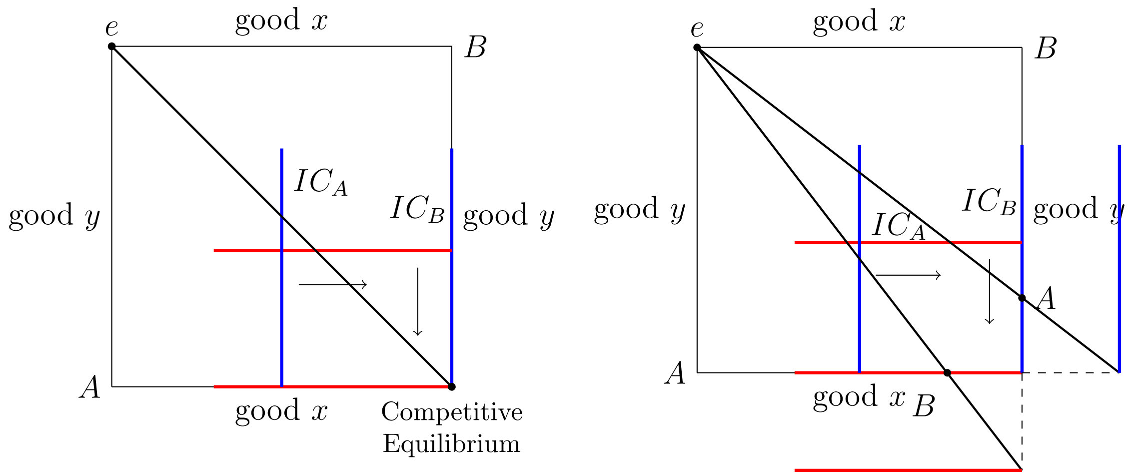
Figure 3.1: Equilibrium in an Edgeworth Box

Suppose now that prices are not equal, so that $p_A > p_B$, or $p_A < p_B$. We have the situation in Figure 2b. Point B is not an equilibrium because agent B wants to push down her indifference curve as maximum as possible, and the allocation in A does not clear the market. The same for point A. Therefore, in this economy, there is only one equilibrium, namely where prices are equal.

3.3 Normative Analysis: the Welfare Theorems

Since the beginning of political economy as an autonomous discipline, one of the most widely argued topics regarded the alleged efficiency, or not, of the markets as the best way to allocate resources among individuals. Only in the late XIXth and XXth century a formal argument was provided to discuss market efficiency, namely that of **Pareto Efficiency** (named after Vilfredo Pareto). On the notion of Pareto Efficiency, properly formalized in the 1950s, two important results have been established and proved, the First and the Second Theorem of Welfare. Roughly speaking, they link Competitive Equilibrium and Pareto Optimality, albeit in different ways. Indeed, the first welfare theorem states that equilibrium outcomes are efficient. The second welfare theorem states that efficient outcomes are competitive equilibria, given the correct prices and endowments.

Definition 3.3.1. An allocation $(x^i)_{i\in\mathcal{I}} \in \mathcal{R}_+^{I\times L}$ is **feasible** if $\sum_{i\in\mathcal{I}} x_l^i \leq \sum_{i\in\mathcal{I}} e_l^i$, for all $I \in \mathcal{L}$.



(a) Unique Equilibrium

(b) Price disequilibrium

Figure 3.2: Two different situations

Definition 3.3.2. Given an economy $\mathcal{E}$, a feasible allocation x is **Pareto Optimal** if there is not other feasible allocation $\hat{x}$ such that, for all $i \in \mathcal{I}$ $u^i(\hat{x}^i) \geq u^i(x^i)$, and for at least one i $u^i(\hat{x}^i) > u^i(x^i)$.

These can be represented graphically through an Edgeworth Box.

In Figure 3a it is shown a not Pareto Optimal Allocation. Indeed for agent A, the point of intersection with the other agent's indifference curve is not optimal since she can easily move toward a higher indifference curve. In figure 3b it is shown a Pareto Optimal Allocation, where the two indifference curves meet.

In general, we can represent the set of all Pareto Allocations in the Edgeworth Box as the line intersecting all the Pareto Optimal allocations, called **contract curve**.

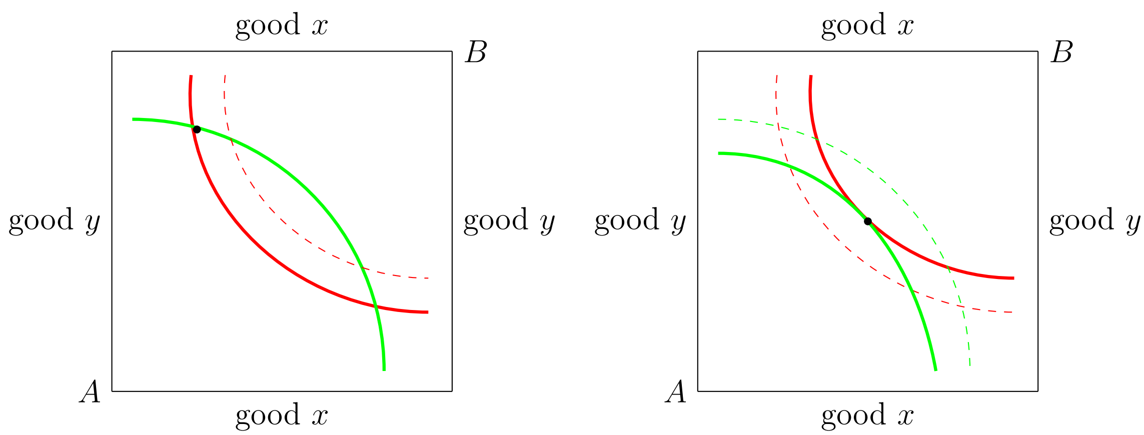
Notice that Pareto Optimality does not say anything about "fairness." Indeed, a Pareto Optimal allocation can be one where one agent has all, or almost all, the endowments, and the other agents have zero or close to zero. Then, according to the definition of Pareto Optimality, an attempt to redistribute wealth should not be optimal.

3.3.1 The First Theorem of Welfare

The first result that links Pareto Optimality and Competitive Equilibrium is the first Welfare Theorem.

Theorem 3.3.1 (First Welfare Theorem)**.** *Suppose an economy that satisfies assumption A2. An equilibrium allocation $x \in \mathbb{R}_+^{I\times L}$ associated with equilibrium price $p \in \mathbb{R}_+^L$ is Pareto Efficient.*

Proof. Suppose that a competitive equilibrium allocation is not Pareto Efficient. Then, it exists $\hat{x}$ such that, for all i, $u^i(\hat{x}^i) \geq u^i(x^i)$ and the inequality is strict for some agents.



(a) A not Pareto Optimal Allocation (b) A Pareto Optimal Allocation

Figure 3.3: Pareto Optimality

By revealed preferences, we can write:

$$p \cdot \hat{x}^i \geq p \cdot x^i$$

For all i, and

$$p \cdot \hat{x}^j > p \cdot x^j$$

for j. Summing inequalities across the agents yields:

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}^i > \sum_{i \in \mathcal{I}} p \cdot x^i = \sum_{i \in \mathcal{I}} p \cdot e^i$$

This because x is an equilibrium allocation, so $\sum_{i \in \mathcal{I}} p \cdot x^i = \sum_{i \in \mathcal{I}} e^i$. Rearranging these inequalities, we have:

$$\sum_{i \in \mathcal{I}} p \cdot \hat{x}_l^i = \sum_{i \in \mathcal{I}} \sum_{i \in \mathcal{L}} p_l \cdot \hat{x}_l^i = \sum_{i \in \mathcal{L}} p_l \sum_{i \in \mathcal{I}} \hat{x}_l^i > \sum_{i \in \mathcal{L}} p_l \sum_{i \in \mathcal{I}} e_l^i$$

and therefore:

$$\sum_{i \in \mathcal{L}} p_l \Big[\sum_{i \in \mathcal{I}} (\hat{x}_l^i - e_l^i) \Big] > 0$$

Since p$\geq$ 0, there must be at least one l such that:

$$\sum_{i \in \mathcal{I}} \hat{x}_l^i > \sum_{i \in \mathcal{I}} e_l^i$$

But then, the society's consumption is greater than the society's endowment. Therefore, we have a contradiction. $\square$

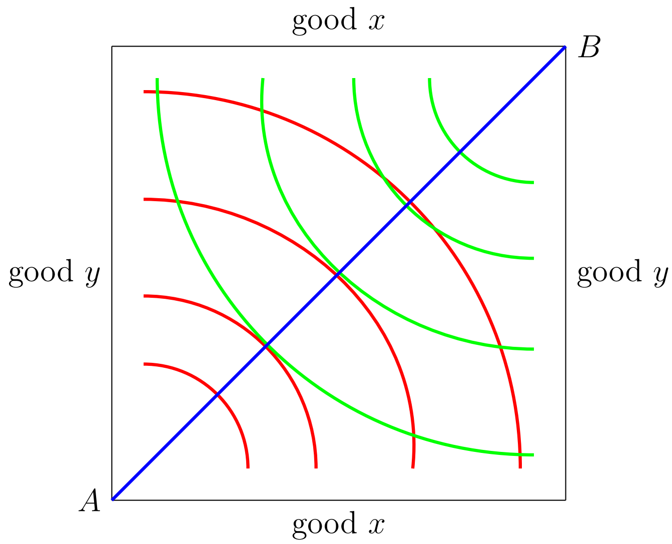


Figure 3.4: The Contract Curve

A way of generalizing the first welfare theorem is through a concept derived from cooperative game theory, namely that of **Core**. The simple idea is that any subset of agents cannot do better alone than a competitive allocation. To see this argument formally, we first need to define the idea of coalitions and **blocking coalitions**.

Definition 3.3.3. Let a set of agents $\subset \mathcal{I}$ denote a coalition of consumers. Then, S blocks allocation $(x^i)_{i \in \mathcal{I}}$ if there is an allocation y such that:

1. $\sum_{i \in S} y^i = \sum_{i \in S} e^i$
2. $u^i(y^i) \geq u^i(x^i)$ for all $i \in S$, with at least one strict preference.

Therefore S is called a **blocking coalition** of allocation $(x^i)_{i \in \mathcal{I}}$.

The intuition is that if a coalition, i.e., a subset of agents, can do better by simply "walking away" and allocating their endowment according to y^i instead of x^i, then x^i cannot be an equilibrium, and therefore, it is not Pareto Optimal.

To see the formal statement and proof, a further definition is required, that of **Core**.

Definition 3.3.4. The **core** of an exchange economy is the set of all unblocked allocations.

Therefore, we have the following result.

Theorem 3.3.2. *Fix an economy $\mathcal{E}$, a Walrasian equilibrium allocation is a core allocation, i.e., it is not blocked by any coalition $S \subseteq \mathcal{I}$.*

Proof. Suppose that x^* is a competitive equilibrium allocation, and it is blocked by some coalition $S \subseteq \mathbb{I}$. Then, there exists an allocation y_i such that:

1. $\sum_{i \in S} y^i = \sum_{i \in S} e^i$
2. $u^i(y^i) \geq u^i(x^i)$ for all $i \in S$, with at least one strict preference.

By revealed preferences:

$$p \cdot y^i \geq p \cdot x^*$$

for all $i \in S$, and:

$$p \cdot y^i > p \cdot x^*$$

for at least one i. Summing up these inequalities, we have:

$$\sum_{i \in S} p \cdot \hat{x}^i > \sum_{i \in S} p \cdot x^* = p \sum_{i \in S} x^*$$

And:

$$\sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot e^i$$

This implies that:

$$\sum_{i \in S} e^i > \sum_{i \in S} x^*$$

so that:

$$p \cdot \sum_{i \in S} (e^i - x^*) > 0$$

Since $p > 0$, this means that for some agents j $p \cdot x^j < p \cdot e^j$, so that x^j cannot be optimal, and therefore not a competitive equilibrium. We have reached a contradiction. □

3.3.2 The Second Theorem of Welfare

So far, it seems that this result establishes the superiority of a free-market, "invisible-hand" market process over any attempt to politically enforce social justice through redistribution. However, although strong, this result is not unique and cannot be generalized too much. Indeed, a second general result says that, under some conditions, any Pareto efficient allocation can be achieved as a competitive equilibrium after some adjustment in agents' endowments. This result is the Second Theorem of Welfare, which can be interpreted as the "converse" of the first theorem.

Theorem 3.3.3. *Given an economy that satisfies assumptions $A1 - A4$, if an endowment allocation e is Pareto optimal, then there exists a price vector $p \in \mathbb{R}^L_+$ such that this is the Competitive equilibrium of the economy.*

Proof. The main idea of the proof can be represented graphically in the following figure:

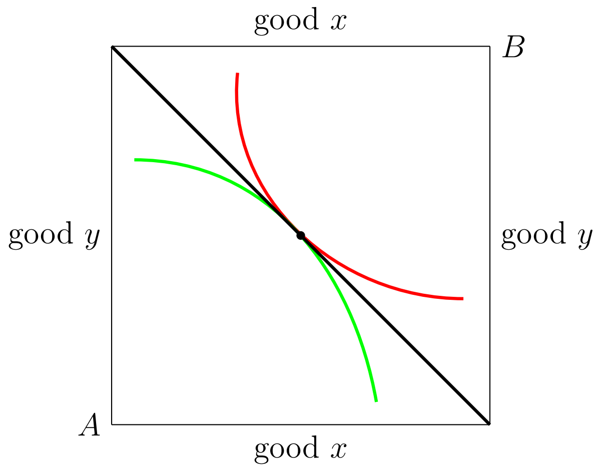


Figure 3.5: A competitive equilibrium

That is, given a competitive equilibrium (where the agents' indifference curves intersect), we can find a price hyperplane passing through it (in this case, where $L = 2$, the hyperplane is a line).[2]

Given its importance, we divide this theorem into four main steps.

1. For each consumer i, we define her upper contour set at her endowment e^i, that is:

$$A^i = \left\{ x \in \mathbb{R}^L_+ : u^i(x^i) > u^i(e^i) \right\}$$

Since $u(\cdot)$ is assumed to be concave, then A^i is a convex set. Furthermore, we can define the "upper contour set of the entire economy" as the Minkowski sum of A^i[3] as:

$$A = \sum_{i \in \mathcal{I}} A^i = \left\{ x \in \mathbb{R}^L : \text{it exists } x^i \in A^i, \text{for each } i \text{ with } x = \sum_{i \in \mathcal{I}} x^i \right\}$$

which is also a convex set.

[2] To do so, we use a powerful mathematical theorem called the **Supporting Hyperplane Theorem**. This states that:

Theorem 3.3.4. *Suppose $A \subset \mathbb{R}^n$ is convex and $x \notin Int(A)^C$, then there exists a vector $p \in \mathbb{R}^L$ such that $p \cdot x \geq p \cdot y$ for every $y \in A$.*

[3] Given two sets, their Minkowski sum is their element-wise sum. For example:

$$A = \{(0,0),(1,0)\} \quad B = \{(0,1),(2,2)\}$$

Then:

$$A + B = \{(0,1),(2,2),(1,1),(3,2)\}$$

2. The total endowment of the economy belongs to the boundary of A^i. By definition of A, $\sum_i e^i \in A$. Further, suppose that $\sum_i e^i \in Int(A)$. Since $Int(A)$ is open, it exists an $\epsilon-$neighbrorhood $N_\epsilon(\sum e_i) \subseteq Int(A)$. Then, there exists an allocation $x \in \mathbb{R}_+^{L\times I}$ and $\lambda < 1$ such that $\sum_i x^i = \lambda \sum_i e^i \in A$. That is:

$$u^i(x) \geq u^i(e^i), \quad \forall i = 1, \ldots, l$$

But then $\frac{1}{\lambda}x$ is feasible, since $\sum x^i = \lambda \sum e^i$ and $\frac{1}{\lambda}\sum x^i = \sum e^i$. By monotonicity:

$$u^i(\frac{x^i}{\lambda^i}) > u^i(x^i)$$

so we have defined a feasible allocation that beats e, which has been assumed to be Pareto optimal. So, we have reached a contradiction.

3. We have a point on the boundary of a convex set. By the Supporting Hyperplane Theorem, there exists a price vector $p \neq 0$ such that:

$$p \cdot y \geq p \cdot \sum_i e^i$$

for any $y \in A$. However, we need to show that $p > 0$. Suppose that $p_l < 0$, for some good l. We can construct an allocation x_k such that $x^i_k = e^i_k$ for $k \neq i$ and every i, and $x^i_l > e^i_l$. By monotonicity, $x^i \in A^i$, so that $\sum_i x^i \in A$, but $p \cdot \sum_i x^i \leq p \cdot \sum_i e^i$. But this contradicts the Supporting Hyperplane theorem above.

4. Now, we want to show that (p, e) is an equilibrium. We need to show that:

$$e^i \in \arg\max_{x \in B^i(p)} u^i(x)$$

Suppose that for consumer i, there is an x^i such that $u_i(x^i) > u^i(e^i)$. We want to show that x^i is not feasible, namely that $p \cdot x^i > p \cdot e^i$. If $u^i(x^i) > u^i(e^i)$, by continuity, there exists a $\lambda \in (0, 1)$ such that $u^i(\lambda x^i) > u^i(e^i)$ for $\lambda < 1$ but close enough to 1. Hence $\lambda x_i + \sum_{j\neq i} e^j \in A$, and therefore, by the Supporting Hyperplane Theorem, we can write:

$$p \cdot \lambda x^i \geq p \cdot e^i$$

Dividing both sides by λ, yields:

$$p \cdot x_i \geq \underbrace{\frac{1}{\lambda} p \cdot e^i > p \cdot e^i}_{\text{since } p>0 \text{ and } e \gg 0}$$

Then, x^i is not feasible.

This completes the proof. $\square$

The two welfare theorems are extremely important results in discussing the normative properties of economies, as well as in assessing different policies. However, they are results extremely general and do not take into account many features of real economies. For instance, both theorems rule out externalities, assume symmetric information, and complete markets. Finally, in the Second Welfare Theorem, nothing is said about how the prospective social decision-maker can know what the preferences of agents are.

3.4 The Existence Theorem

The fundamental step in the analysis of general equilibrium is to show that, given the mathematical properties we have used so far to characterize a pure-exchange economy, Pareto optimal allocations, and competitive equilibrium, the latter actually exists. As seen, for many decades after Walras' work, this was considered less important with respect to comparative statics and normative analysis. However, things changed in the 1940s and 1950s, and successful attempts to provide formal proof of existence were published in 1954 (Arrow and Debreu 1954; McKenzie 1954). These results followed, with slight modifications, a technique previously used by John Nash to show the existence of the equilibrium solution named after him. In particular, they rested on a fixed-point argument. In this section, given its importance, this result will be proved carefully, following the result by Arrow & Debreu. First, a less general theorem with stronger assumptions. Then, the general result.

Theorem 3.4.1 (Arrow-Debreu Existence Theorem (1954))**.** *Let $\mathcal{E}$ be an economy satisfying the following assumptions:*

A.1 For all agents $i \in \mathcal{I}, u^i(x)$ is continuous

A.2 For all agents $i \in \mathcal{I}$, $u^i(x)$ is increasing, namely $u^i(x) > u^i(x')$ if $x \gg x'$

A.3 For all agents $i \in \mathcal{I}$, $u^i(x)$ is concave

A.4 For all agents $i \in \mathcal{I}, e^i \gg 0$

Then, a Competitive Equilibrium exists.

Notice how powerful this result is. Indeed, it says that a competitive equilibrium always exists, given conditions $A1 - A4$, which are very general. In particular, it does not depend on any specific functional form of utility functions, assuming that this is continuous, increasing, and concave.

3.4.1 Existence under strong assumptions

Before entering into the classical proof, it can be useful to assume some stronger assumptions. Then, we can replace $A2 - A3$ with the following:

(A.2') Strict monotonicity: $Du'(x) \gg 0, \forall x$

(A.3') Strict concavity of $u(\cdot)$

This relaxation has two implications. First, prices equal to zero are ruled out for any $l = 1, 2, \dots, L$. Further, from $A3'$, the demand correspondence is a function.

Therefore, the existence theorem can be rewritten as follows:

Theorem 3.4.2. *Let $\mathcal{E}$ be an economy satisfying assumption $A1 - A2' - A3' - A4$. Then a Competitive Equilibrium exists.*

Then, we can introduce a further notion, that of **excess demand function**.

The excess demand function

Definition 3.4.1. The **excess demand function** of agent i is:

$$z^i(p) = x^i(p, p \cdot e^i) - e^i$$

where $x^i(\cdot)$ is the Marshallian demand. The **aggregate excess demand** is:

$$z(p) = \sum_{i \in \mathcal{I}} z^i(p)$$

Where $z : \mathbb{R}^L_+ \to \mathbb{R}^L$

From this definition, it is clear that if there is a price such that $z(p) = 0$, then this price is an equilibrium price. Indeed, into the notion of excess demand function enters each agent's Marshallian demand, which is the solution to the Utility Maximization Problem, and further, since $z(p) = 0$, all markets clear. Therefore, proving the existence of a Competitive equilibrium boils down to showing the existence of a solution to $z(p) = 0$. Since $z(p)$ is the main object we are working with from now on, some of its properties must be established.

Proposition 15. *$z(p)$ has the following properties:*

1. *$z(p)$ is continuous*
2. *$z(p)$ is homogenous of degree zero*
3. *$p \cdot z(p) = 0$ for all p*
4. *There is an $s > 0$ such that $z_l(p) > -s$ for every l and every p*

5. *If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then $max\{z_1(p^n), \dots, z_L(p^n)\} \to \infty$.*

Proof.

1. This property simply derives from the fact that $x_i(p, p \cdot e^i)$, the Marshallian Demand, is continuous in p. And the sum of a continuous function is a continuous function too

2. Again, this derives from the Marshallian Demand. Indeed, if all the prices change by a factor λ, then the Marshallian Demand does not change (i.e., it is Homogeneous of Degree Zero). This property, as well as the one above, is preserved under summation (whereas $\sum_i e^i$ does not count since it is a constant)

3. Notice that:
$$p_1 z_1(p) + p_2 z_2(p) + \dots p_n z_n(p) =$$
$$p \cdot \Big[\sum_i x_i(p, p \cdot e^i) - \sum_i e^i \Big] =$$
$$\sum_i [p \cdot x(p, p \cdot e^i) - p \cdot e^i] = 0$$
Indeed, by Walras' Law $p \cdot x = p \cdot e^i$, therefore, all above equals to zero. It is interesting to notice a point. If we have n-goods, it is sufficient to clear the market for $n-1$-goods, then, also the market for the n^{th}-good clear.

4. Since $x^i \in \mathbb{R}^L_+$, then $z^i(p) \geq -e^i$. This is because the lowest value the Marshallian Demand can take is 0 ($x \in \mathbb{R}^2_+$). So in the case of all demands equal to 0, $z(p)$ is given just by the total initial endowment.

5. This comes from the strictly increasing utility of each good. As some, but not all, prices go to zero, there must be some consumers whose wealth is not going to zero. Because she has strongly monotone preferences, she must demand more and more of one of the goods whose price is going to zero.[4]

$\square$

The intuition when $L = 2$

Let's see now the proof of the existence under strong assumptions. First, recall that, from Walras's Law, $p \cdot z(p) = 0$, but $z(p) = 0$ holds only in equilibrium. This is what we need to prove. By Walras' law, if $L - 1$ markets clear, then also the L^{th} market clears.

Let's start with $L = 2$. Since $z(p)$. Since $z(p)$ is homogeneous of degree zero, we can normalize $p_2 = 1$ so that the price vector is $p = (p_1, 1)$. In equilibrium $z_1(p) = z_2(p) = 0$, but by Walras' Law, we only need to show that $z_1(p_1, 1) = 0$. Further, we know that $z_1(p_1, 1)$ is continuous on a connected domain. We want to show that:

[4] is a requirement needed to keep track of the possibility of Giffen Goods (recall that the Marshallian Demand does not necessarily satisfy the Law of Demand).

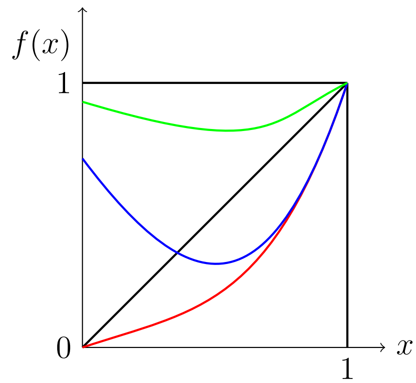


Figure 3.6: $f : [0, 1] \to [0, 1]$

1. when p_1 is small enough $z_1 > 0$: that is, the preferences are strongly monotone

2. when p_1 is big enough $z_1 \in [-Z, 0]$: that is $e^i > 0$.

Therefore, we can apply the **intermediate value theorem**, so that there must exist at least on p_1^* such that $z_1(p_1^*, 1) = 0$. Thus, p_1^* is a competitive equilibrium.

For (1), assume $p_1 \to 0$, so that the price of good 1 is very cheap. Then $z_1(p_1, 1)$ or $z_2(p_1, 1)$ go to infinity. Suppose $z_2(p_1, 1) \to \infty$. Then $p_2 \cdot z_2(p_1, 1) = z_2(p_1, 1) \to \infty$. And we know that $p_1 \cdot z_1(p_1, 1) > -p_1 s$, for some fixed s. Hence, Walras' Law, $p_1 \cdot z_1 + p_2 \cdot z_2 = 0$ would be violated for some sufficiently small p_1. So $z_1(p_1, 1) \to \infty$.

For 2 assume that $p_1 \to \infty$. By homogeneity of degree zero, we can write $p_2' = \frac{1}{p_1} \to 0$, and normalize the price of $p_1 = 1$. Again, since $z_1 + p_2' \cdot z_2 = 0$, this implies $z_2 \to \infty$ and $z_1 \to 0$.

The proof under strong assumptions

When $L > 2$, we cannot use the **intermediate value theorem**, but we need a more general result. This is the **Brouwer Fixed Point Theorem**.

Theorem 3.4.3 (Brouwer Fixed Point Theorem)**.** *Given a* ***compact*** *and* ***convex*** *set $A \subset \mathbb{R}^n$ and a* ***continuous*** *function $f : A \to A$, then there exists a fixed point $x \in A$ such that $f(x) = x$*

Proof. Assume the special case when $A = [0, 1]$. Figure 6 shows very simply that it is impossible to have a continuous function from the closed interval $[0, 1]$ into itself that does not cross at least once the 45 degree line, i.e. the line of points where $f(x) = x$.

This renders a simple image of the theorem. Let's enter more in some (simple) details. Let's define:

$$g(x) = f(x) - x$$

Therefore, we can write:

$$g(0) = f(0) - 0 \geq 0$$

and

$$g(1) = f(1) - 1 \leq 0$$

But then, by the **intermediate value theorem**, we know that there exists (at least) an x for which $g(x) = 0$. Then:

$$g(x) = f(x) - x = 0$$
$$f(x) = x$$

$\square$

Let's see the main proof now. Take a price vector p. Since $z(p)$ is Homogeneous of Degree Zero, then we can normalize p so that we multiply all the prices by $\frac{1}{\sum_i p_l}$:

$$p = \begin{pmatrix} \frac{p_1}{\sum p_l} \\ \vdots \\ \frac{p_L}{\sum p_l} \end{pmatrix}$$

Such that $\sum_{l=1} p_l = 1$ (and therefore, knowing $L-1$ prices, makes it possible to know also the L^{th} price). This allows us to say that prices belong to the unit simplex with $L-1$ dimensions, i.e.:

$$\Delta^{L-1} = \left\{ p \in R_+^L : \sum_{l=1}^{L} p_l = 1 \right\}$$

Notice that Δ^{L-1} is a convex and compact set.

We can define the following function, $g : \Delta^{L-1} \to \Delta^{L-1}$ which maps its domain (in this case the unit simplex) into itself:

$$g(p_l) = \frac{p_l + \max\{0, z_l(p)\}}{1 + \sum_l \max\{0, z_l(p)\}}$$

Notice that $\sum_l g(p_l) = 1$, so we are still in the simplex. And that $g(p_l)$ is a continuous function. Thus, we can apply the conditions of the Brouwer Theorem.

Intuitively, we can see that $g(p_l) > p_l$ if there is excess of demand of l (so there is no market clearing). But we know that, by Brouwer, there exists for sure a p^* such that:

$$p^* = g(p^*)$$

implies

$$g(p_l^*) = \frac{p_l^* + \max\{0, z_l(p^*)\}}{1 + \sum_l \max\{0, z_l(p^*)\}} = p_l^*$$

for all $l \in L = 1, \ldots, L$.

Now we show that p^* must clear all markets. A way to see this is by cross-multiplying and rearranging terms:

$$p_l^* \sum \max\{0, z_l(p^*)\} = \max\{0, z_l(p^*)\}$$

Multiply both sides by $z_l(p^*)$

$$z_l(p^*)p_l^* \sum \max\{0, z_l(p^*)\} = z_l(p^*)\max\{0, z_l(p^*)\}$$

Sum over L:

$$\sum_l z_l(p^*)p_l^* \Big[\sum \max\{0, z_l(p^*)\}\Big] = \sum_l z_l(p^*)\max\{0, z_l(p^*)\}$$

Still, note that for property 3 of $z(p)$ all the left-hand term of the equation above goes to zero. So then, we have:

$$\sum_l z_l(p^*)\max\{0, z_l(p^*)\} = 0$$

The proof is close to the ending. Notice that each member of the linear combination above is larger or equal to zero since it is either 0 or a square $[z_l(p)^2]$. But from the equation above, we know that $z_l(p^*)$ must be less or equal to zero. Assume $p^* > 0$. Then, $z_i(p^*)$ must be equal to 0, otherwise, Walras' Law is violated. If $p^* = 0$, it can be possible that $z_l(p^*) < 0$. But then some $z_k(p^*)$ must be unbounded, therefore contradicting the property 5 of the excess demand function. Therefore, we cannot have $p^* = 0$, and $z_l(p^*) = 0$ for all $l = 1, \ldots, L$:

$$p_1^* \cdot z_1(p^*) + \ldots p_L^* \cdot z_L(p^*) = 0$$

This means that there exists a price vector p^* for which the excess demand function for all goods is equal to 0. This concludes the proof.

3.4.2 Existence: the classical proof

We can now prove the equilibrium existence under the general assumptions:

A.1 For all agents $i \in \mathcal{I}, u^i(x)$ is continuous

A.2 For all agents $i \in \mathcal{I}$, $u^i(x)$ is increasing, namely $u^i(x) > u^i(x')$ if $x \gg x'$

A.3 For all agents $i \in \mathcal{I}$, $u^i(x)$ is concave

A.4 For all agents $i \in \mathcal{I}, e^i \gg 0$

Here is a general outline of the main intuition behind the classical proof, following Arrow & Debreu's classical paper (Arrow and Debreu 1954) The main difference with the previous proof is that the excess demand function is not necessarily single-valued anymore, but it is a correspondence. Then, instead of looking for a price vector that solves $z(p) = 0$, we are going to define a map Ψ that takes the set of price-aggregate demand pairs (p, x) into itself. This map is going to be defined as follows: given a price-aggregate demand pair, agents optimize given prices, and a new aggregate demand is obtained. The demand is a correspondence, and we establish that is a non-empty, convex-valued, and upper hemi-continuous correspondence of prices. Then, to obtain new prices, we allow for a "price-player" additional agent to take the old aggregate demand as given and set price: her choice correspondence is an upper hemi-continuos, convex-valued and non-empty correspondence of aggregate demand. If the old prices are equilibrium prices, then the price player won't change prices.

Therefore, all the choices have been put together into a correspondence Ψ that maps a price vector and an aggregate demand into new prices (chosen by the price player) and a new aggregate demand (chosen by the agents). Then, applying a fixed-point theorem for correspondences, the Kakutani fixed-point theorem, we argue that a fixed point of our map corresponds to a Competitive equilibrium.

Before starting, let's recall (without proof) the two fundamental mathematical results that will be employed in the proof.

Theorem 3.4.4 (Kakutani's Fixed Point Theorem)**.** *Suppose $A \subset \mathbb{R}^n$ is a convex, closed, and bounded set. Suppose $f : A \rightrightarrows A$ is a convex-valued, non-empty for all $x \in A$ and upper hemi-continuous correspondence. Then there exists $x \in A$ such that $f(x) = x$.*

Theorem 3.4.5 (Maximum's Theorem)**.** *Suppose we have a continuous correspondence $f : X \times \Phi \to \mathbb{R}$, with $X \subseteq \mathbb{R}^n, \Theta \subseteq \mathbb{R}^m$, and a correspondence $\Gamma : \Theta \to X$ compact valued and continuous. Let $v : \Theta \to \mathbb{R}$ be the value function:*

$$v(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta)$$

and $D : \Theta \rightrightarrows X$ such that:

$$D(\theta) = \arg\max_{x \in \Gamma(\theta)} f(x, \theta)$$

Then:

- *$D(\cdot)$ is not empty*
- *$D)\cdot)$ is compact-valued*
- *$D(\cdot)$ is upper hemi-continuous*
- *$v(\cdot)$ is continuous*

We can start now to prove the theorem through a series of different steps.

Step 1: normalize prices and reformulate the consumer's problem

We need to normalize $p \in \Delta^{L-1}$ in order to avoid $p = 0$ (namely, that some goods are free). Otherwise, $B^i(p, p \cdot e^i)$ may be not compact. Define:

$$T = \{x \in \mathbb{R}^L_+ : x \leq 2 \cdot e\}$$

where $e = \sum_i e^i$. Further, we define each agent i's constrained demand correspondence:

$$\Psi_i(p) = \underset{x \in B^i(p) \cap T}{\arg\max}\, u^i(x)$$

Since $B^i(p) \cap T$ is compact and continuous, and u is continuous, then by Berge's Maximum Theorem, we have $\psi^i(p)$ is not empty-valued, compact-valued and upper-hemi continuous. Furthermore, since $u^i(x)$ is quasi-concave, $\Psi(p)$ is convex-valued.

Step 2: Introducing the "price player"

We now introduce a "Price Player," namely a fictitious player who sets the prices to maximize the value of the aggregate excess demand (at the old prices). Roughly speaking, every consumer takes prices as given and by solving the utility maximization problem, finds her demand; instead, the Price Player takes the demand as given and chooses the prices.

We define the Price Player as P. So her problem is:

$$\Psi^P(x^1, \ldots, x^l) = \underset{p \in \Delta^{L-1}}{\arg\max}\, p \cdot \left(\sum_{i \in \mathcal{I}} x^i - \sum_{i \in \mathcal{I}} e^i \right)$$

Where $\sum_{i \in \mathcal{I}} x^i - \sum_{i \in \mathcal{I}} e^i$ is the aggregate excess demand z. This is linear in prices and continuous in p. By Maximum Theorem, $\Psi^P : T \times T \times \ldots T \to \Delta^{L-1}$ is not-empty, compact-valued and convex-valued.

Further, if the excess demand is strictly positive, then we can normalize the prices, setting the highest prices = 1 and other prices equal to 0.

Step 3: A fixed-point exists

Define:

$$\Psi : \Delta^{L-1} \times T \times T \cdots \times T \rightrightarrows \Delta^{L-1} \times T \times T \cdots \times T$$

as:

$$\Psi(p, x^1, x^2, \ldots, x^l) = \Psi^P(x^1, x^2, \ldots, x^l) \times \Psi^1(p) \cdots \times \Psi^l(p)$$

The product of non-empty, convex-valued, and compact-valued upper hemi-continuous correspondences is itself a non-empty, convex-valued, and compact-valued upper hemi-continuous correspondence.

Therefore, we can use the Kakutani Fixed Point Theorem, and therefore there exists a fixed point:

$$(p^*, x^*) \in \Psi(p^+, x^*)$$

where $x^{i*} \in x^i(p)$ for each i.

Step 4: The fixed point is actually a competitive equilibrium

The last step is to show that (p^*, x^*) is actually a Competitive Equilibrium, namely that it solves the Utility Maximization Problem for each agent in the economy and clears all the markets.

In particular, we need to show that:

1. At (p^*, x^*) the aggregate excess demand is:

$$z^* = \sum_{i=1}^{n} x^{i*} - e \leq 0$$

Suppose not. Let $z_l^* = \max\{z_k^*\} > 0$. Then the price player must set p^* such that $p^* \cdot z^* = z_l^* \geq 0$. However, the Budget constraint $p \cdot (x^i - e^i) \leq 0$ implies:

$$p^* \cdot \left(\sum_i x_i^* - \sum_i e^i \right) = p^* \cdot z^* \leq 0$$

then we have reached a contradiction.

2. Individual optimality:

$$x^{i*} \in \arg\max_{x \in B^i(p^*)} u^i(x)$$

By the previous point, we know that:

$$x^{i*} \leq \sum_{j=1}^{n} x^{j*} \leq e \leq 2 \cdot e$$

Suppose x_i^* is not optimal given p^* without constraint T. Another way of seeing it is that we know $x^* \in \Psi^i(p) = \arg\max x \in B^i(p) \cap T u^i(x)$. We want to show that $x^* \in \arg\max_{x \in B^i(p^*)} u^i(x)$ (which is a smaller set than $B^i(p) \cap T$)). Take $\hat{x}^i \in B^i(p^*)$ such that:

$$u^i(\hat{x^i}) > u^i(x^{i*})$$

Since $B^i(p^*)$ is convex-valued, then:

$$\lambda x^{i*} + (1 - \lambda)\hat{x}^i \in B^i(p^*)$$

for all $\lambda \in [0, 1]$. Further, because $u^i(x)$ is concave:

$$u^i(\lambda x^{i*} + (1 - \lambda)\hat{x}^i) \geq \lambda u^i(x^{i*}) + (1 - \lambda)u^i(x^{i*}) > u^i(x^{i*})$$

For $\lambda \approx 1$, $\lambda x^{i*} + (1 - \lambda)\hat{x}^{i*} \in B^i(p^*) \cap T$. This is a contradiction to the optimality of $x^{i*} \in B^i(p^*) \cap T$.

3. The last step is to show that the markets clear. By 1 we know that:

$$z^* = \sum_{i=1}^{n} x^{i*} - e \leq 0$$

By Walras' Law, we have:

$$p^* \Big(\sum_{i=1}^{n} x^{i*} \Big) = p^* \cdot e$$

so that the rice player's value function is zero. If $z_i \leq 0$, for some l, then:

$$p_l^* = 0$$

This means that we can simply give these excess goods to, say, agent 1 and get markets clear to maintain individual optimality.

This concludes the proof. We have shown that in every pure exchange economy, given very general conditions, there always exists a Competitive Equilibrium.

Appendix : Stochastic Orders and Order Statistics

Introduction

To study lotteries with monetary payoffs, we can compare utility functions, or instead, we can compare payoff distributions. To the latter, it is related to the notion of stochastic orders. Intuitively, there are ways according to which random outcomes can be compared: according to the level of returns, namely if a distribution $F(\cdot)$ yields always higher returns than $G(\cdot)$; or according to the dispersion of returns, namely $F(\cdot)$ being always less risky than $G(\cdot)$.

To begin with, let's recall some useful definitions:

Definition .0.2. The support of p is the set of all values such that $p(x_i) > 0$. That is:

$$supp(p) = \Big\{x_i : p(x_i) > 0\Big\}$$

Example .0.1. *Take a fair die. The* ***support*** *is the set* $S = \{1, 2, 3, 4, 5, 6\}$*, and each value of the set has a probability of occurrence of* $\frac{1}{6}$*.*

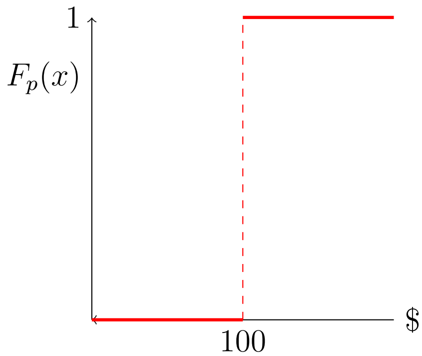
Definition .0.3. The **Cumulative Distribution Function** (CDF) $F(x) = \mathbb{P}[X \leq x]$ is the probability of the event $X \leq x$.

Notice that: $F(x)$ is non decreasing, and $\sum_i^n \mathbb{P}[X \leq x_i] \neq 1$.

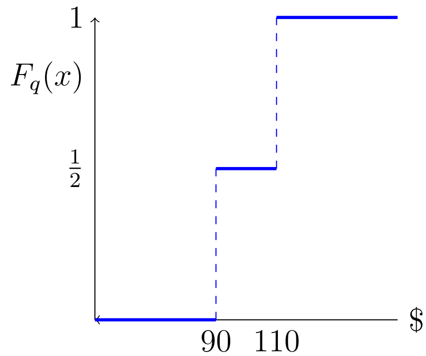
Example .0.2. *Let's continue with the example of the fair die. Take* $x = 4$*. Therefore,* $F(4) = \mathbb{P}[X \leq 4]$ *is equal to* $\frac{2}{3}$*. If* $x = 3$*, then* $F(3) = \frac{1}{2}$ *and so on...*

Example .0.3. *Let's see another example. Consider the following problem. A decision maker with an income of* 100 *must decide if to accept or reject a bet of* 10\$ *on a fair coin toss. There are two lotteries* p *and* q*, which correspond to respectively reject or accept the bet, with the following distributions:*

$$F_p(x) = \begin{cases} 0 & \textit{if } x < 100 \\ 1 & \textit{if } x \geq 100 \end{cases} \quad \textit{and} \quad F_q(x) = \begin{cases} 0 & \textit{if } x < 90 \\ \frac{1}{2} & \textit{if } 90 \leq x < 110 \\ 1 & \textit{if } x \geq 110 \end{cases}$$



(a) The CDF of lottery p



(b) The CDF of lottery q

Figure 7: CDFs of a discrete distribution

These lotteries can be conveniently represented by the graph of the corresponding CDFs. If the decision maker rejects the bet, the probability that her income becomes less than 100 *equals* 0 *and that her income is at least* 100 *equals* 1*. If the decision maker accepts the bet, the probability that her income is less than* 90 *is* 0*, the probability that her income is at least* 90 *and less than* 110 *is* $\frac{1}{2}$*, and the probability that her income is at least* 110 *is* 1[5]*.*

Example .0.4. *Let's see the following lotteries:*

$$p = \begin{cases} 0 & \textit{with probability } 0 \\ 100 & \textit{with probability } \frac{1}{2} \\ 200 & \textit{with probability } \frac{1}{2} \end{cases}$$

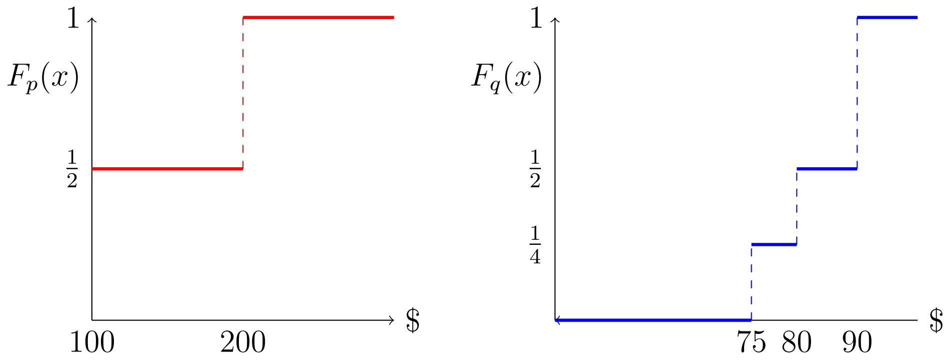
and

$$q = \begin{cases} 0 & \textit{with probability } \frac{1}{4} \\ 75 & \textit{with probability } \frac{1}{4} \\ 80 & \textit{with probability } \frac{1}{4} \\ 90 & \textit{with probability } \frac{1}{4} \end{cases}$$

The graph of the CDFs are:

From the graph (a), $\mathbb{P}[x \leq 100]$ *is equal to* 0*,* $\mathbb{P}[x \leq 100] = \frac{1}{2}$*, and the probability of obtaining at most* 200 *is* 1*. For (b), the probability of obtaining at most* 75 *is* $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$*, at most* 80 *is* $\frac{1}{2} + 14 = \frac{2}{3}$*, and at most* 90 *is* 1*.*

[5]The reasoning is the following: if you accept the bet, you start with 100 but have probability $\frac{1}{2}$ (the coin is fair) of ending up with 90. Similarly, you have $\frac{1}{2}$ of ending up with 110. Therefore $\frac{1}{2} + \frac{1}{2} = 1$



(a) The CDF of lottery p (b) The CDF of lottery q

Figure 8: CDFs of the distribution

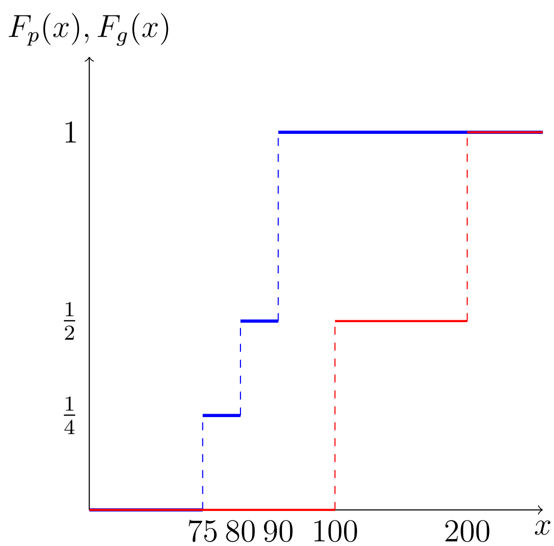
First-Order Stochastic Dominance

Given a pair of lotteries, one can ask when a lottery *always* generates higher utility than another, given that the decision-maker is an expected utility maximizer. To answer this question, we need a concept of stochastic dominance, namely the **First-Order Stochastic Dominance**.

Definition .0.4. Given two Cumulative Distribution Functions F and G, we say that F **First order stochastically dominates** G if, for all $x \in [0, 1]$, we have:

$$F(x) \leq G(x)$$

To have a graphical example, see the following graph:



This refers to the lotteries described in the example 1.4. Intuitively, p is better than q since you can always obtain a higher payoff with a higher probability. This can be seen by looking at the CDFs of the two distributions represented in the figure. The distribution of p (red) is always below to the distribution of q (blue). Therefore:

$$F_p(x) \leq F_q(x)$$

and:

$$p \succeq_{FOSD} q$$

An important result establishes the relationship between the definition above and the expected value associated with each distribution, assuming that $u : [0,1] \to \mathbb{R}$ is differentiable and increasing.

Theorem .0.6. *Take a differentiable function $u : [0,1] \to \mathbb{R}$. Then, these two statements are equivalent:*

1. *$F(x) \geq G(x)$, namely $F(x)$ first-order stochastically dominates $G(x)$*
2. *$\mathbb{E}_F[u(x)] \geq \mathbb{E}_G[u(x)]$*

Proof. We can write:

$$\begin{aligned} \mathbb{E}_F[u(x)] - \mathbb{E}_G[u(x)] = & \int_0^1 u(x)dF(x) - \int_0^1 u(x)dG(x) = \\ & \int_0^1 u(x) \underbrace{F'(x)}_{f(x)} dx - \int_0^1 u(x) \underbrace{G'(x)}_{g(x)} dx = \end{aligned}$$

Using integration by parts and rearranging:

$$u(1) \cdot F(1) - u(0) \cdot F(0) - \int_0^1 u'(x)F(x)dx - u(1) \cdot G(1) - u(0) \cdot G(0) - \int_0^1 u'(x)G(x)dx =$$

$$u(1) \underbrace{[F(1) - G(1)]}_{\equiv 0} - u(0) \underbrace{[F(0) - G(0)]}_{\equiv 0} - \int_0^1 u'[F(x) - G(x)]dx =$$

$$-\int_0^1 u'[F(x) - G(x)]dx$$

The last expression is greater than zero because $F(x) - G(x) < 0$ by FOSD, $u'(\cdot) > 0$. Therefore:

$$\mathbb{E}_F[u(x)] \geq \mathbb{E}_G[u(x)]$$

□

This result is quite powerful because it is just assumed that $u(\cdot)$ is increasing.

Second-Order Stochastic Dominance

FOSD involves the idea of "higher/better" versus "lower/worse." Another widely used notion of stochastic order involves a comparison based on the relative riskiness of lotteries.

Given two distributions with $F(x)$ and $G(x)$ with the same mean $\mathbb{E}_F[x] = \mathbb{E}_G[x]$ (or $\int_0^1 x \cdot f(x)dx = \int_0^1 x \cdot g(x)dx$) we say that $G(x)$ is riskier than $F(x)$ if **every risk averse** decision-maker prefers $F(x)$ to $G(x)$.

Definition .0.5. For any two distributions $F(x)$ and $G(x)$, with the same mean, $F(x)$ **Second-order stochastically dominates** (or is less risky than) $G(x)$ if for every non-decreasing function $u : [0,1] \to \mathbb{R}$, we have:

$$\int_0^1 u(x) \cdot f(x)dx \geq \int_0^1 u(x) \cdot g(x)dx$$

Another way to describe the second-order stochastic dominance involves the notion of **super-cumulative distribution**. A super-cumulative distribution is the integral of the CDF:

$$S(x) = \int_0^1 F(x)dx$$

Therefore, we have $F(x) = S'(x)$ and $S(0) = 0$.

The following result establishes an equivalence between the previous definition of SOSD and the super-cumulative distribution.

Proposition 16. *Consider two distributions $F(\cdot)$ and $G(\cdot)$ with the same mean. Then the following statements are equivalent:*

1. $F(\cdot) \succeq_{SOSD} G(\cdot)$
2. $S_F(x) = \int_0^1 F(x)dx \leq S_G(x) = \int_0^1 G(x)dx$

Proof. We want to show that:

$$\underbrace{\int_0^1 u(x)f(x)dx}_{U(l_F)} - \underbrace{\int_0^1 u(x)g(x)dx}_{U(l_G)} \geq 0$$

Notice that:

$$\int_0^1 u(x)f(x)dx =$$

$$\underbrace{u(1)F(1) - u(0)F(0) - \int_0^1 u'(x)F(x)dx}_{\text{by integration by parts}} =$$

$$\underbrace{u(1) - \int_0^1 u'(x)F(x)dx}_{\text{since } u(0)F(0)=0} =$$

by integrating by parts another time, we have:

$$u(1) - u'(1)\underbrace{\int_0^1 F(t)dt}_{S(1)} - u'(0)\underbrace{\int_0^1 F(t)dt}_{0} - \int_0^1 u''(x)\underbrace{\int_0^1 F(x)dx}_{S(x)}\,dx =$$

$$u(1) - \Big[u'(1)S(1) - \int_0^1 u''(x)S(x)dx\Big]$$

Therefore:

$$\int_0^1 u(x)f(x)dx - \int_0^1 u(x)g(x)dx =$$

$$u(1) - u'(1)S_F(1) - \int_0^1 u''(x)S_F(x)dx - u(1) - u'(1)S_G(1) - \int_0^1 u''(x)S_G(x)dx$$

rearranging we have:

$$u'(1)\cdot[S_G(1) - S_F(1)] - \int_0^1 u''(x)\Big[S_F(x) - S_G(x)\Big]dx$$

Notice that:

$$S(1) = \int_0^1 F(x)dx \equiv \int_0^1 F(x)\cdot 1dx =$$

integrating by parts

$$F(1)\cdot 1 - F(0)\cdot 0 - \int_0^1 f(x)\cdot xdx$$

$$F(1) - \int_0^1 f(x)\cdot xdx =$$

$$1 - \mathbb{E}(x)$$

Since $F(\cdot)$ and $G(\cdot)$ have the same mean, then:

$$1 - \mathbb{E}_F(x) - 1 + \mathbb{E}_G(x) = 0$$

Therefore:

$$\int_0^1 u(x)f(x)dx - \int_0^1 u(x)g(x)dx =$$
$$-\int_0^1 u''(x)\Big[S_F(x) - S_G(x)\Big]dx \geq 0$$

Since $u(\cdot)$ is concave, then $u''(\cdot) < 0$ and $S_F(x) \leq S_G(x)$.

$\square$

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