Microeconomics
General Economic Equilibrium

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1 General Equilibrium with I consumers, J firms and L commodities

Let's define an economy as a system composed by consumers $i = 1, \dots, I$, firms $j = 1, \dots J$ and Commodities $l = 1, \dots L$, where $I, J, L \in R_+^m$

Some important preliminary definitions are the following:

Definition 1.1 (Feasible Allocations)**.** An allocation (x, y) specifies a consumption vector $x_i \in X_i$ for any i and a production vector for any j. An allocation is feasible if:

$$\sum_i x_{Ii} \leq \bar{\omega}_l + \sum_j y_{lj} \quad \forall l$$

Definition 1.2 (Pareto Optimality)**.** A feasible allocation (x, y) is Pareto Optimal if there is no other feasible allocation (x', y') such that $x'_i \succeq_i x_i$ for all i and $x'_i \succ_i x_i$ for some i.

Definition 1.3 (Price Equilibrium with Transfers)**.** Given an economy specified by $(\{X_i, \succeq_i)\}_{i=1}^I, \{Y_j\}_{j=1}^J, \bar{\omega}\})$, an allocation (x^*, y^*) and a price vector $p = (p_1, \dots, p_L)$ constitute a Price Equilibrium with Transfers (PEwT) it there is an assignment of wealth levels $(w_1, \dots w_l)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:

1. for every firm j, y_j^* maximizes profits in Y_j. I.e. $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.
2. for every consumer i, x_i^* is the maximal element for $\succeq_i$ in the budget set $\{x_i \in X_i : p \cdot x_i \leq w_i\}$
3. $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

Roughly speaking, conditions 1 and 2 say that firms and consumers are respectively profit and utility maximizers. Instead, condition 3 is the market clearing condition: the sum of all individual demands equal the sum of all individual supplies plus the total endowment of the economy.

A less general condition than PEwT is that of the Walrasian Equilibrium in Private Ownership Economies. We define a Private Ownership economy as a system where consumers have claims to a share of the profits of the firm j, $\theta_{ij} \in [0,1]$ (where $\sum_i \theta_{ij} = 1$ and $\sum_j \theta_{ij} \in [0, J]$).

Definition 1.4 (Walrasian Equilibrium in a Private Ownership Economy)**.** Given an economy specified by $(\{X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \{\omega_i, \theta\ :_{i1}, \cdots, \theta_{iJ}\}_{i=1}^{I})$, an allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ constitute a Walrasian Equilibrium it there is an assignment of wealth levels $(w_1, \ldots w_l)$ with $\sum_i w_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$ such that:

1. for every firm j, y_j^* maximizes profits in Y_j. I.e. $p \cdot y_j \leq p \cdot y_j^*$ for all $y_j \in Y_j$.
2. for every consumer i, x_i^* is the maximal element for $\succeq_i$ in the budget set $\{x_i \in X_i : p \cdot x_i \leq p \cdot \omega_i + \sum_j \theta_{ij}(p \cdot y_j^*)\}$
3. $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

1.1 Welfare Theorems

From the definitions above one can state, and therefore prove, an important result of theoretical economics, namely that every price equilibrium with transfers (and therefore, any Walrasian equilibrium) is Pareto Optimal. This is the First Fundamental Theorem of Welfare Economics.

Theorem 1.1 (The First Fundamental Theorem of Welfare Economics)**.** *If preferences $\succeq_i$ are Locally Non Satiable and if (x^*, y^*, p) is a Price Equilibrium with Transfers (PEwT) then, the allocation (x^*, y^*) is Pareto Optimal. In particular, any Walrasian Equilibrium is Pareto Optimal.*

Proof. Suppose (x^*, y^*, p) is a PEwT with the following wealth levels $(w_1, \ldots, w_l)$ and $\sum_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$.

By the condition 2 in the definition of PEwT, we know that each consumer maximizes her utility. Put in another way:

$$x_i \succ x_i^* \Rightarrow p \cdot x_i > \omega_i \tag{1}$$

Or, in words, if there exists a bundle which is preferred to the optimal bundle, then it is not in the budget set.

Since $\succeq_i$ are Locally Non Satiated, then:

$$x_i \succeq x_i^* \Rightarrow p \cdot x_i \geq \omega_i \tag{2}$$

Let's prove (2), which can be rewritten as follows: If $x_i > x_I^* \Rightarrow p \cdot x_i > \omega_i$ and preferences are Locally Non Satiatied, then $x_i \succeq x_i^* \Rightarrow p \cdot x_i \geq \omega_i$.

To see this, suppose that the condition above does not hold. Then, we write:

$$x_i \succeq x_i^* \Rightarrow p \cdot x_i \leq \omega_i$$

In words, x_i belongs to the Budget Set. But then, since preferences are Locally Non Satiated, there exists a x_i' such that given $||x_i' - x_i|| < \epsilon$, $x_i' \succ_i x_i$ and x_i' is feasible (i.e. $x_i' \cdot p < \omega_i$)

But preferences $\succeq_i$ are transitive, so:

$$x_i' \succ_i x_i \succeq_i x_i^* \Rightarrow x_i' \succ x_i^*$$

This contradicts the optimality of x_i^*.

Let's go back now to the main proof. Consider now an allocation (x_i, y_i) that Pareto Dominates (x^*, y^*). I.e. $x_i \succeq x_i^* \;\; \forall i$ and there is at least one consumer for which $x_i \succ x_i^*$ We want to show that such an allocation is not feasible, and therefore the equilibrium allocation is Pareto Optimal.

By (2):

$$p \cdot x_i \geq \omega_i \quad \forall i$$

and by (1):

$$p \cdot x_i > \omega_i$$

for at least some i. Then

$$\sum_i p \cdot x_i > \sum_i \omega_i = p \cdot \bar{\omega} + \sum_j p \cdot y_j^*$$

From condition 1 in the definition of PEwT we know that firms are profit maximizers. So y_j^* is the quantity of input that maximizes profits. So:

$$p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$$

Then we have:

$$\sum_i p \cdot x_i > p \cdot \bar{\omega} + \sum_j p \cdot y_j^* \geq p \cdot \bar{\omega} + \sum_j p \cdot y_j$$

And therefore, by transitivity:

$$\sum_i p \cdot x_i > p \cdot \bar{\omega} + \sum_j p \cdot y_j$$

This means that (x, y) is not feasible. Then (x^*, y^*) is Pareto Optimal.

□

Note that the assumption of Locally Non Satiation of preferences is necessary. Furthermore, we are assuming no externalities.

Theorem 1.2 (Second Fundamental Theorem of Welfare Economics)**.** *Consider an economy specified by* $(\{X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\omega}\})$ *and suppose that every* Y_i *is convex and every* $\succeq$ *is convex and locally non satiated. Then, for every Pareto Optimal allocation* (x^*, y^*) *there is a price vector* $p = (p_1, \ldots, p_L) \neq 0$ *such that* (x^*, y^*, p) *is a price quasi-equilibrium with transfers.*

Proof. Let's start by defining, for every agent, the set of consumptions preferred to x_i^*:

$$V_i = \{x_i \in X_i : x_i \succ_i x^*\}$$

Define also the set of aggregate consumption bundles:

$$V = \sum V_i = \left\{ \sum_i x_i \in R^L : x_1 \in V_1 \ldots x_I \in V_I \right\}$$

and the aggregate production set:

$$Y = \sum_j Y_j = \left\{ \sum_j \in R^L : y_1 \in Y_1, \ldots y_j \in Y_J \right\}$$

V can be split into I individual consumptions, each preferred by its corresponding consumer to x_i^*. Instead of Y we can take $Y + \{\bar{\omega}\}$ which is the aggregate production set with its origin shifted to $\bar{\omega}$.

Let's divide the proof in several steps:

1. Note that V_i is convex (since preference are convex). Then even V and Y are convex since the sum of each finite number of convex sets is itself convex.

2. Notice that $V \cap (Y + \{\bar{\omega}\}) = \emptyset$. Indeed if there is a vector in both sets, then, with the given endowments and technologies it would be possible to produce an aggregate vector that could be used to give every consumer i a consumption bundle preferred to x_i.

3. Still since V and $Y + \{\bar{\omega}\}$ are both convex, we can see that there is a price different to zero and a number r such that $p \cdot z \geq r$ for every $z \in V$, and $p \cdot z \leq r$ for every $z \in Y + \{\bar{\omega}\}$ (This is the Separating Hyperplane Theorem for Convex Sets).

4. If $x_i \succeq_i x_i^*$ for every i, then $p \cdot (\sum_i x_i) \geq r$.

5. $p \cdot (\sum_i x_i^*) = p \cdot (\bar{\omega} + \sum_j y_i^*) = r$

6. For every j, we have $p \cdot y_j \leq p \cdot y_i^*$ for all $y_j \in Y_j$ For any firm j and $y_j \in Y_j$ we have $y_j + \sum_{h \neq j} y_h^* \in Y$. Then:

   $$p \cdot \left(\bar{\omega} + y_j + \sum_{h \neq j} y_h^* \right) \leq r = p \cdot \left(\bar{\omega} + y_j + \sum_{h \neq j} y_h^* \right)$$

   Hence, $p \cdot y_j \leq p \cdot y_j^*$

7. For every i, if $x_i \succ x_i^*$, then $p \cdot x_i \geq p \cdot x_i^*$. Then, considering any $x_i \succ x_i^*$ we have:

$$p \cdot \left(\bar{\omega} + x_i + \sum_{k \neq i} x_k^* \right) \leq r = p \cdot \left(\bar{\omega} + x_i + \sum_{k \neq i} x_k^* \right)$$

   Hence, $p \cdot x_i \leq p \cdot x_i^*$

8. Since step 7 and 8 fulfill conditions ii) and iii) in the definition Price Equilibria with Transfers, and (x^*, y^*) is feasible, then the proof is complete.

□

1.2 Existence of an Equilibrium

Let's consider the issue of the existence of the equilibrium in a simplified framework, that of pure exchange economy. This is characterized by continuous, strictly convex and strongly monotone $\succeq_i$ and $\sum_i \omega_i >> 0$.

The existence of an equilibrium means the existence of a price vector p that supports the Pareto optimal allocatio (x^*, y^*) and that clears all the L markets simultaneously.

To do so, we need to define two further objects, namely the Excess Demand Function of agent i, $z_i(p)$ and the Aggregate Demand Function $z(p)$. For each agent the EDF is:

$$z_i(p) = x_i(p, p \cdot \omega_i) - \omega_i$$

Instead, the Aggregate EDF is:

$$z(p) = \sum_i z_i(p) = \sum_i x_i(p, p \cdot \omega_i) - \sum_i \omega_i$$

More in detail, then, we can interpret $z(p)$ as a vector with L components, one for each commodities, that defines the excess demand in any market for any good. The Market clearing condition requires that $z(p)$ is a null vector, i.e. that markets clear for all of the $1, \ldots, L$ goods

In the existence of equilibrium, the key idea is that of using some properties of $z(p)$ in order to set up the conditions for using a fixed point technique (in the simplest case, the Brouwer Fixed Point Theorem, see below). Therefore, it is important to outline, as well as to discuss briefly, some the properties of $z(p)$.

1. $z(p)$ is continuous. This property simply derives from the fact that $x_i(p, p \cdot \omega_i)$, the Walrasian Demand, is continuous in p. And the sum of continuous function is a continuous function too.

2. $z(p)$ is homogenous of degree zero. Again, this derives from the Walrasian Demand. Indeed, if all the prices change by a factor k, then Walrasian Demand does not change (i.e. it is Homogenous of Degree Zero). This property, as well as the

one above, is preserved under summation (whereas $\sum_i \omega_i$ does not count, since it is a constant)

3. $p \cdot z(p) = 0$ for all p. Notice that:

$$\Rightarrow p \cdot \left[\sum_i x_i(p, p \cdot \omega_i) - \sum_i \omega_i \right] = 0$$
$$\sum_i [p \cdot x(p, p \cdot \omega_i) - p \cdot \omega_i] = 0$$

But by Walras' Law $p \cdot x = w$, therefore, all above equals to zero. It is interesting to notice a point. If we have n-goods, it is sufficient to clear the market for $n-1$-goods, then, also the market for the n^{th}-good clear.

4. There is an $s > 0$ such that $z_l(p) > -s$ for every l and every p. This because the lowest value the Walrasian Demand can take is 0 ($x \in R^2_+$). So in the case of all demands equal to 0, $z(p)$ is given just by the total initial endowment.

5. If $p^n \to p$, where $p \neq 0$ and $p_l = 0$ for some l, then $\max\{z_1(p^n), \ldots, z_L(p^n)\} \to \infty$. This is a requirement needed to keep track of the possibility of Giffen Goods (recall that the Walrasian Demand does not necessarily satisfy the Law of Demand).

Then, it is possible to state and demonstrate the theorem for the existence of an equilibrium. Notice that existence means that there is at least one solution, but still notice that there could be more than one.

Theorem 1.3 (Existence of an Equilibrium)**.** *Let the function $z(p)$ satisfy the properties above. Then the system $z(p)$ has a solution. That is, a Walrasian equilibrium exists in a pure exchange economy in which $\sum_i \omega_i >> 0$ and every consumer has continuous, strictly convex and strongly monotone preferences.*

Proof. Take a price vector $\hat{p}$:

$$\hat{p} = \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_l \end{pmatrix}$$

Since $z(p)$ is Homogenous of Degree Zero, then we can normalize $\hat{p}$ so that we multiply all the prices by $\frac{1}{\sum_i p_l}$:

$$\hat{p} = \begin{pmatrix} \frac{\hat{p_1}}{\sum \hat{p}_l} \\ \vdots \\ \frac{\hat{p_L}}{\sum \hat{p}_l} \end{pmatrix}$$

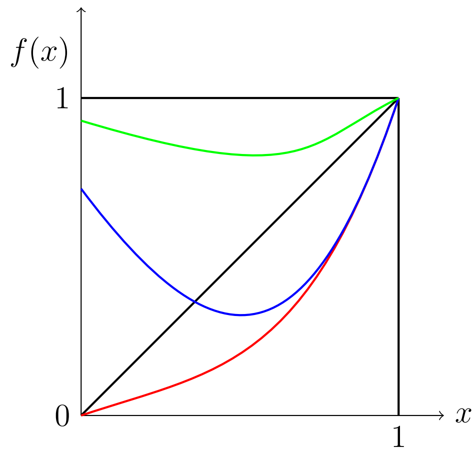

Figure 1: $f : [0,1] \to [0,1]$

Such that $\sum_{l=1} p_l = 1$ (and therefore, knowing $L-1$ prices, makes it possible to know also the L^{th} price). This allows use to say that prices belong to the unit simplex with $L-1$ dimensions, i.e.:

$$S^{L-1} = \left\{ p \in R^L_+ : \sum_{l=1}^{L} p_l = 1 \right\}$$

To show that there is a vector $p \in S^{L-1}$ that clear all the markets, we use a Fixed Point Technique, in particular the Brouwer Fixed Point Theorem.

Theorem 1.4 (Brouwer Fixed Point Theorem)**.** *Let $f : X \to X$, f continuous, X compact. (i.e. a function that maps X into itself) Then, there exists an $x \in X$ such that $f(x) = x$.*

Proof. Figure 1 shows very simply that it is impossible to have a continuous function from the closed interval $[0,1]$ into itself that does not cross at least once the 45 degree line, i.e. the line of points where $f(x) = x$.

This renders a simple image of the theorem. Let's enter more in some (simple) details. Let's define:

$$g(x) = f(x) - x$$

Therefore, we can write:

$$g(0) = f(0) - 0 \geq 0$$

and

$$g(1) = f(1) - 1 \leq 0$$

But then, by the Intermediate Value Theorem (see figure 2) we know that there exists (at least) an x for which $g(x) = 0$. Then:

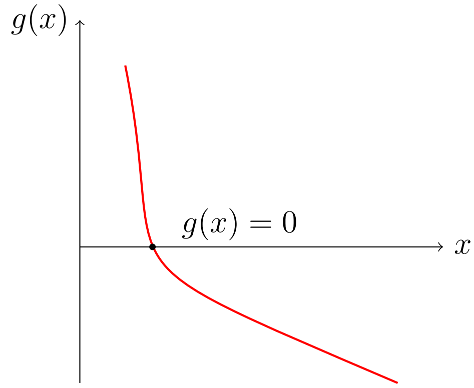

Figure 2: A simple graphical representation of the Intermediate Value Theorem

$$\begin{aligned} g(x) &= f(x) - x = 0 \\ f(x) &= x \end{aligned}$$

□

Let's now go back to the main proof.

We can define the following function, $g : S^{L-1} \to S^{L-1}$ which maps its domain (in this case the unit simplex) into itself:

$$g(p_l) = \frac{p_l + \max\{0, z_l(p)\}}{1 + \sum_l \max\{0, z_l(p)\}}$$

Notice also that $\sum_l g(p_l) = 1$. So we are still in the simplex. And that $g(p_l)$ is a continuous function so that we can apply the conditions of the Brouwer Theorem.

Intuitively, we can see that $g(p_l) > p_l$ if there is excess of demand of l (so there is no market clearing). But we know that, by Brouwer, there exists for sure a p^* such that:

$$p^* = g(p^*)$$

implies

$$g(p_l^*) = \frac{p_l^* + \max\{0, z_l(p^*)\}}{1 + \sum_l \max\{0, z_l(p^*)\}} = p_l^*$$

for all $l \in L = \{1, \dots, L\}$.

A way to see this is cross-multiplying and rearranging terms:

$$p_l^* \sum \max\{0, z_l(p^*)\} = \max\{0, z_l(p^*)\}$$

Multiply both sides by $z_l(p^*)$

$$z_l(p^*) p_l^* \sum \max\{0, z_l(p^*)\} = z_l(p^*) \max\{0, z_l(p^*)\}$$

Sum over L:

$$\sum_l z_l(p^*)p_l^* \left[\sum \max\{0, z_l(p^*)\} \right] = \sum_l z_l(p^*)\max\{0, z_l(p^*)\}$$

Still note that for property 3 of $z(p)$ all the left-hand term of the equation above goes to zero. So then, we have:

$$\sum_l z_l(p^*)\max\{0, z_l(p^*)\} = 0 \tag{3}$$

The proof is close to the ending. Notice that each member of the linear combination above is larger or equal to zero since it is either 0 or a square $[z_l(p)^2]$. But from equation 3 above, we know that $z_l(p^*)$ must be less or equal to zero. However, since $p^* \cdot z(p^*) = 0$ and $p^* > 0$ (prices cannot be negative or equal to zero), then $z_l(p^*) = 0$, for all $l \in L = \{1, \ldots, 0\}$.

Or, written in the extensive form:

$$p_1^* \cdot z_1(p^*) + \ldots p_L^* \cdot z_L(p^*) = 0$$

This means that exists a price vector p^* for which the excess demand function for all goods is equal to 0. This concludes our proof. □