Microeconomics
Notes on Classical Demand Theory

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1 The Expenditure Minimization Problem

The *expenditure minimization problem* (EMP) is the dual of the *Utility Maximization Problem* (UMP). Then, when $p >> 0$ and $u > u(0)$:

$$\begin{aligned} \min_{x \geq 0} \quad & p \cdot x \\ \text{s.t.} \quad & u(x) \geq u \end{aligned} \tag{1}$$

Whereas the UMP was about the maximum amount of x needed to maximize utility, under a budget constraint, the EMP instead is about the minimum amount of expenditure needed to reach a definite level of utility. Therefore, the optimal bundle x^* is the bundle which solve the EMP, that is, that minimize $p \cdot x$ subject to a utility constraint. In other words *to solve the EMP means to seek the minimum amount the consumer must spend at price p to get for himself utility level u.*

Geometrically, it is the point of the set $\{x \in R^l_+ : u(x) \geq u\}$ which lies on the least possible budget line associated to a definite price vector (see Figure 1).

As the EMP is the dual of the UMP, the following result makes apparent the relationship between them.

Proposition 1 (3.E.1 (MWG)**.** *Suppose $u(.)$ is a continuous utility function representing $\succeq$ L.N.S. and defined on $X = R^l_+$, and that price vector is $p >> 0$. Then:*

1. *If x^* is optimal in the UMP when $w > 0$, then x^* is optimal too in the EMP, when the required utility level is $u(x^*)$, and the minimized expenditure level in this EMP is w*

2. *If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility level in this UMP is u*

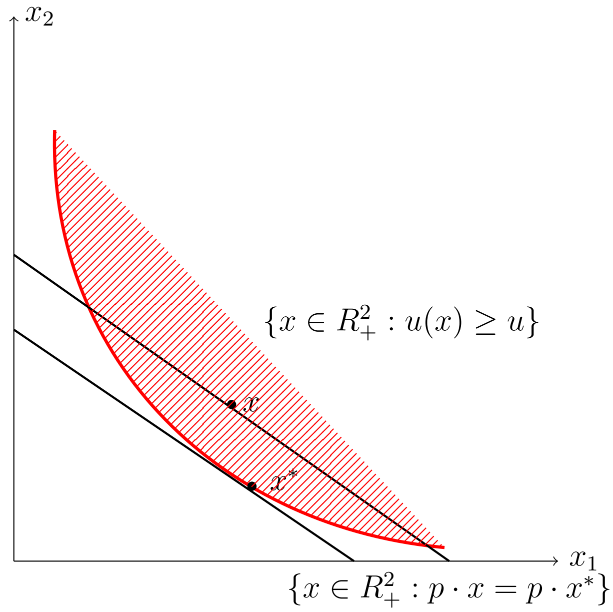

Figure 1: The Expenditure Minimization Problem

Proof. 1. We can show this by contradiction. Assume x^* is not optimal in the EMP. Then, it exists a x' such that $u(x') > u(x^*)$, and $p \cdot x < p \cdot x^* \leq w$. Still, however, $\succeq$ are LNS, so that we can find an x'' such that $u(x'') > u(x')$ and $p \cdot x' < w$. But then $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$. This contradicts the assumption of x^* being optimal in the UMP. Finally, since x^* solves the UMP when prices are p, then $p \cdot x^* = w$.

2. Since $u > u(0)$, then $x^* > 0$ and $p \cdot x^* > 0$. Suppose x^* is not optimal. Then there exist a $x' > x^*$ such that $u(x') > u(x^*)$ and $p \cdot x' < p \cdot x*$. Take a bundle $x'' = \alpha x'$ (with $\alpha \in (0,1)$). By continuity of $u(.)$, if $\alpha \sim 1$, then $u(x") > u(x^*)$ and $p \cdot x" < p \cdot x^*$. But this contradicts the optimality of x^* in the EMP. Then x^* must be optimal in the UMP when $w = p \cdot x$ and $\max_x u = u(x^*)$.

□

Note finally that a solution to the EMP exists always under very general conditions: the constraint set must be non empty.

2 The Expenditure Function

The value of the EMP can be determined by the function $e(p,u)$, called the *expenditure function*. Its value for any pair (p,u) is simply $p \cdot x^*$, where x^* is the solution to the EMP. Thus $e(p,u)$ is the minimum expenditure required to achieve utility u at prices p. A way of writing this is the following:

$$e(p,u) = \min\{p \cdot x : U(x) \geq u, x \geq 0\}$$

The basic properties of this function are given in the following proposition.

Proposition 2 (3.E.2 (MWG)**.** *Suppose $u(.)$ is a continuous utility function representing $\succeq$ L.N.S. and defined on $X = R^l_+$. Then $e(p,u)$ is:*

1. *Homogenous of Degree one in p*
2. *Strictly increasing in u and non-decreasing in p_l, $\quad \forall l$*
3. *Concave in p*
4. *Continuous in p and u*

Proof. 1. To see that $e(p,u)$ is HDZ in p note that in the EMP, if p change, utility is unaffected. In other words, the EMP now becomes: min $\alpha \cdot x$ subjcet to $u(x) \geq u$. If x^*, then $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p,u)$.

2. $e(p,u)$ being not strictly increasing in u means that if u increases, then the value of $e(p,u)$ does not. To see this, assume x' and x'' as optimal consumption bundles for the utility levels $u(x')$ and $u(x'')$, where $u(x'') > u(x')$ and $p \cdot x'' < p \cdot x'$. Take a bundle $\hat{x} = \alpha x''$ (with $\alpha \in (0,1)$). By continuity of $u(.)$, if $\alpha \sim 1$, then $u(\hat{x}) > u(x')$ and $p \cdot \hat{x} < p \cdot x'$. But then, x' is not optimal in the EMP.

   Let's see now $e(p,u)$ being not decreasing in princes. This means that when p decrease, $e(p,u)$ does not. Assume p'' and p', where $p''_l \geq p'_l$ and $p''_k = p'_k \forall l \neq k$. Let x'' be an optimizing consumption bundle in the EMP for prices p''. Then $e(p'',u) = p'' \cdot x'' \geq p' \cdot x' = e(p',u)$

3. To see concavity, assume $\bar{u}$ and $p'' = \alpha p + (1-\alpha)p'$, (with $\alpha \in [0,1]$). Suppose x^* is optimal in the EMP, at prices p''. Then:

$$\begin{aligned}
&e(p'',\bar{u}) = p'' \cdot x^* = \\
&[\alpha p + (1-\alpha)p']x^* = \\
&\alpha p \cdot x^* + (1-\alpha)p' \cdot x* \geq \\
&\geq \alpha e(p,\bar{u}) + (1-\alpha)e(p',\bar{u})
\end{aligned} \tag{2}$$

   Whereas: $\alpha p \cdot x^* + (1-\alpha)p'x^* \geq \alpha e(p,\bar{u}) + (1-\alpha)e(p',\bar{u})$, $\alpha e(p,u^*) * (1-\alpha)e(p',u^*)$, and $u^* = u(x^*) > \bar{u}$

□

Intuitively, the meaning of concavity is simply that if there is an optimal consumption bundle in the EMP, whose value is given by $e(p,u)$, if p changes, so that the new price vector is p', then the upper bound of the new consumption bundle is given by $p' \cdot x$, a linear transformation of $p \cdot x$.

Note, finally, that there is a strong relation between the *Expenditure Function* $e(p, u)$ and the *Indirect Utility Function* $v(p, w)$. That is, for any $p >> 0$, $w > 0$ and $u > u(0)$, then:

- $e(p, v(p, w)) \equiv w$
- $v(p, e(p, u)) \equiv u$

These conditions mean that for a fixed price vector p, $e(p, .)$ and $v(p, .)$ are one the inverse of the other.

3 The Hicksian Demand Function

The set of optimal consumption bundles in the EMP is known as the *Hicksian Demand Correspondence* (or Function, if univalued), defined as $h(p, u) \in R^l_+$.

$$h(p, u) = \arg\min_x \sum_{i=1}^{n} p \cdot x \quad s.t. \;\; u(x) \geq x \tag{3}$$

In other words, $h(p, u)$ is the *set of consumption bundles that the consumer would purchase at prices p is she wished to minimize her expense but still achieve utility u.*

Then, exactly like the *Walrasian Demand*, $x(p, w)$ is the solution to the UMP, at given (p, w), $h(p, u)$ is the solution to the expenditure minimization problem at given (p, u).

The Hicksian Demand has three basic properties:

Proposition 3 (3.E.3 (MWG))**.** *Suppose $u(.)$ is a continuous utility function representing $\succeq$ L.N.S. and defined on $X = R^l_+$. Then, for any $p >> 0$ $h(p, u)$ is:*

1. *Homogenous of Degree Zero in p*
2. *No excess utility: $\forall x \in h(p, u), u(x) = u$*
3. *Convexity/uniqueness: if $\succeq$ is convex, then $h(p, u)$ is a convex set; and if $\succeq$ is strictly convex, so that $u(.)$ is strictly quasi-concave, then there is a unique element in $h(p, u)$.*

Proof. 1. Since the constraint is the same in the EMP for $(\alpha p, x)$ and (p, x), then:

$$\min_{u(x) \geq u} \alpha p \cdot x = \alpha \min_{u(x) \geq u} p \cdot x$$

2. Suppose that there is some $x \in h(p, u)$ such that $u(x) > u \geq u(0)$. Take a bundle $x' = \alpha x$, with $(\alpha \in (0, 1)$. Then $p \cdot x' < p \cdot x$, and since u(.) is continuous (by the Intermediate Value Theorem) there is an α such that $u(x') \geq u$. This contradicts the assumption that $x \in h(p, u)$.

3. Note that $h(p,u) = \{x \in R^l_+ : u(x) \geq u\} \cup \{x : p \cdot x = e(p,u)\}$ is the intersection to two convex sets, and hence is convex. If preferences are strictly convex, $x, x' \in h(p,u)$, then for $\alpha \in [0,1], x'' = \alpha x + (1-\alpha)x' \succ x$ and $p \cdot x'' = e(p,u)$. But this contradicts "no excess utility."

□

We can relate the *Hicksian Demand* and the *Walrasian (or Marshallian) Demand* as follows:

- $h(p,u) \equiv x(p, e(p,u))$ [Recall that $e(p,u) \equiv e(p, v(p,w)) \equiv w$]
- $x(p,w) \equiv h(p, v(p,w))$ [Recall that $v(p,w) \equiv v(p, e(p,u)) \equiv u$]

Another result allows us to link $h(p,u)$ and the *Compensated Law of the Demand*. In a nutshell, demand and prices move in opposite directions for prices changes that are accompanied by *Hicksian Wealth Compensations*. This means that $h_k(p,u)$ is decreasing in p_k, i.e. *Hicksian Demand is always downward sloping*. Note that this is not always true in the case of the *Walrasian Demand* (even if it is typically the case). For example we can find such situations like those involving Giffen Goods (the prices rises and the demand rises too).

Proposition 4 (3.E.4. (MWG))**.** *Suppose $u(.)$ is a continuous utility function representing $\succeq$ L.N.S. and defined on $X = R^l_+$, and $h(p,u)$ is uni-valued, for any $p >> 0$. Then $h(p,u)$ satisfies the Compensated Law of Demand. For all p and p':*

$$(p' - p) \cdot [h(p', u) - h(p,u)] \leq 0 \tag{4}$$

Proof. In the EMP, at prices p, $h(p,u)$ is optimal. This means that it allows the consumer to attain the same level of utility, but with a lesser expenditure. That is:

$$\begin{aligned} p' \cdot h(p', u) &\leq p' \cdot h(p,u) \\ p \cdot h(p,u) &\leq p \cdot h(p,' u) \end{aligned} \tag{5}$$

Subtracting these equations yields the equation (4). Indeed:

$$\begin{aligned} & p' \cdot h(p', u) - p' \cdot h(p,u) - p \cdot h(p,u) + p \cdot h(p,' u) = \\ & (p' - p) \cdot [h(p', u) - h(p,u)] \leq 0 \end{aligned} \tag{6}$$

□

We can see why the *Hicksian Demand* is always downward sloping in Figure 2. The original prices for x_1 and x_2 determine a bundle set $B_{p,w}$ and $h(p,u)$ is h^A. As prices change, to reach the same utility, the new bundle set becomes $B_{p',w}$. So then, the new $h(p,u)$ is h^B, which is still on the Indifference Curve I. This because the *Hicksian Demand* refers to a EMP problem, so then, given an utility level, the rational consumer must find the best way of reaching it.

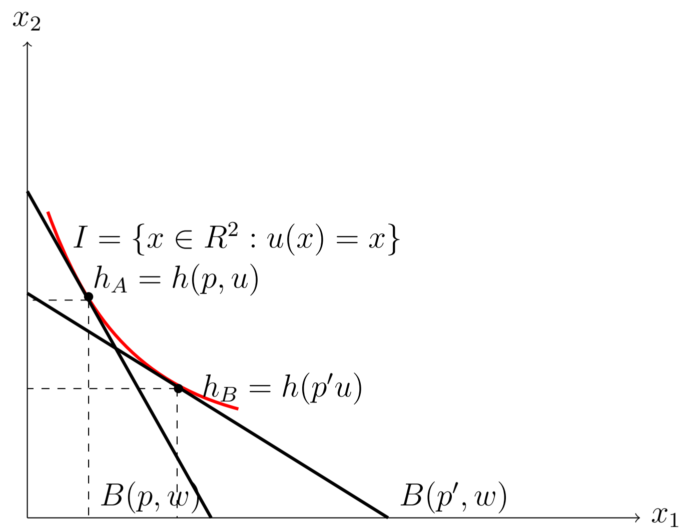

Figure 2: Changes in the Hicksian Demand as prices change

In the *Walrasian Demand* setting, instead the problem is different, since the constraint is the budget set. If prices change, and so does the budget set, the new optimal x^* lies on a different indifference curve.

Finally there is an important result that allows us to recover easily $h_i(p, u)$ from the Expenditure Function $e(p, u)$, called *Shephard's Lemma*.

Proposition 5 (Shephard's Lemma)**.** *Suppose that $u(.)$ is a continuous utility function representing L.N.S. preference $\succeq$ and suppose that $h(p, u)$ is a function. Then, the $e(p, u)$ is differentiable in p, and for all $i = 1 \ldots, n$*

$$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u) \tag{7}$$

Proof. As seen above, $e(p, u)$ is the value function associated to the EMP. Therefore, we can write:

$$\begin{array}{cc} \min\limits_{x \geq 0} & p \cdot x \\ \text{s.t.} & u(x) \geq u \end{array}$$

Taking the Lagrangian of the equation above gives:

$$\mathfrak{L} = p \cdot x + \lambda(u - U(x)) \tag{8}$$

Let's apply now the *Envelope Theorem*[1]. Then we can write the derivative of $e(p, u)$ as being equal to the derivative of $\mathfrak{L}$ for any x^* in $h(p, u)$.

[1]This theorem states the following:

$$\frac{\partial e(p,u)}{\partial p_i} = \frac{\partial \mathfrak{L}}{\partial p_i} = x_i^* \quad \forall x^* \in h(p,u) \tag{10}$$

Where x_i^* is the *Hicksian Demand* for good x_i

□

Let's recap the important identities that link the $e(p,u)$, the $v(p,w)$, the $x(p,w)$ and the $h(p,u)$. Recall that the optimal x^* in the UMP is the same in the EMP. Furthermore:

- $e(p, v(p,w)) \equiv w$ The minimum expenditure necessary to reach utility $v(p,w)$ is m
- $v(p, e(p,u)) \equiv u$. The maximum utility from income $e(p,u)$ is u
- $x_i(p,w) \equiv h_i(p, v(p,w))$ The *Walrasian Demand* at income w is equal to the *Hicksian Demand* at utility $v(p,w)$.
- $h_i(p,u) \equiv x_i(p, e(p,u))$ The *Hicksian Demand* at utility u is the same as the *Walrasian Demand* at income $e(p,u)$.

In particular the last result is important, since it shows that the Hicksian Demand is equal to the Walrasian Demand at the minimum income necessary, at the given prices, to achieve the desired level of utility. Therefore, the *Hicksian Demand is simply the Walrasian Demand function for the various goods if the consumer's income is "compensated" so as to achieve some target level of utility.*

From the identities above it is possible to derive a result similar to the *Shephard's Lemma*, but for Utility Maximization and *Walrasian Demand*: the *Roy's Identity.* This offers a method of deriving the *Walrasian* Demand Function of a good for some consumer from the Indirect Utility Function, $v(p,w)$ of that consumer.

Proposition 6 (Roy's Identity)**.** *Let $u(.)$ be continuous and representing LNS and strictly convex $\succeq$, and $u(.)$ is differentiable. Then:*

$$x_i(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} \quad \textit{for } i = 1, \ldots, k \tag{11}$$

Theorem 3.1 (Envelope Theorem)**.** *For $\Theta \subseteq R$, let $f : X \times \Theta \to R$ be a differentiable function, let $v(\theta) = max_{x \in X} f(x,\theta)$ and let $X^*(\Theta) = \{x \in X : f(x,\theta) = V(\Theta)\}$ If V is differentiable at Θ then, for any $x^* \in X^*(\Theta)$,*

$$V'(\Theta) = \frac{\partial f(x^*, \theta)}{\partial \theta} \tag{9}$$

Roughly speaking this theorem states that, if we change some parameters of the objective, changes in the optimizer do not contribute to the change in the objective function

Proof. We know that if x^* is optimal in the UMP, then it is optimal also in the EMP. Therefore we can write :

$$x(p, w) \equiv h(p, u)$$

at given p, w, u. Furthermore, we know also that:

$$u \equiv v(p, e(p, u))$$

That is, no matter what the prices are, if the consumer has the minimal income to get utility u, at prices p, then the maximal utility is u.

We can differentiate with respect to p and obtain:

$$\frac{\partial v(p, e(p, u))}{\partial p_i} + \frac{\partial v(p, e(p, u)}{\partial p_i} \cdot \frac{\partial e(p, u)}{\partial p_i} = 0$$

Note that:

$$\frac{\partial v(p, e(p, u)}{\partial p_i} \cdot \frac{\partial e(p, u)}{\partial p_i} \equiv \frac{\partial v(p, w)}{\partial w} \cdot \frac{\partial e(p, u)}{\partial p_i}$$

and

$$\frac{\partial e(p, u)}{\partial p_i} \equiv h_i(p, u) \equiv x_i(p, w)$$

These identities hold for all p, w. Therefore, rearranging, we have:

$$\begin{gathered} -\frac{\partial v(p, w)}{\partial w} x_i(p, w) = \frac{\partial v(p, w)}{\partial p_i} = \\ x_i(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} \end{gathered} \tag{12}$$

□

4 The Slutsky Equation

At this point one could question what is the meaning of the results above, and more in general, of all the Consumer Theory in such a mathematical fashion. The object of consumer theory must be to analyze how a rational consumer reacts when he faces some changing in prices and wealth. There are some results (listed below) which describe the rational consumer's behavior with regard to her Walrasian Demand. However, in order to fully assess this point, it is not sufficient to rest upon the UMP. Indeed, the total change can be decomposed in two parts, one that involves the Walrasian Demand, and one that involves the Hicksian Demand.

Note however, that the Hicksian Demand is not directly observable (one of its parameters is u). Still $h(p, u)$ is computable through the Walrasian Demand, which is observable (in principle). We have seen that there are some important results to recover Hicksian Demand and Walrasian Demand from the Expenditure Function and Indirect Utility (i.e. the Shephard's Lemma and the Roy's identity). It is important now to relate $h(p,u)$ and $x(p,w)$ in a more general way, in order to make possible a detailed analysis of how a change in the prices affects the change in the demand.

It is easy to have some intuition on why changes in prices or wealth have some effect on the demand. For what concerns the Walrasian Demand, i.e. the solution to the UMP for all prices and income levels, there are some important results worth to be briefly listed and recapped.

The *Wealth Effects* indicates how the demand changes when wealth changes. This is represented by the following $(1 \times L)$ vector:

$$D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \in R^L \tag{13}$$

The *Price Effects* instead shows how the change of the price of one good affects the demand for all the goods. These effects are represented by the following square matrix:

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} \dots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} \tag{14}$$

Finally, these *Substitution Effects* (i.e. Wealth Effects and Price Effects) can be expressed by the following square matrix called *Slutsky Matrix*:

$$S(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \dots & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} x_L(p,w) \\ \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \dots & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \end{bmatrix} \tag{15}$$

The results above make clear a very simple and intuitive fact. Assume a change in prices, say a raising of p_k. Then the consumer faces two different situations: first, the good k is more expensive relative to other goods, so one can expect a decline in k's consumption, and depending from the relation between k and other goods, it could be the case that even their consumption falls. In any case, there is a "substitution", or "cross-substitution" effect. Second, the consumer's real income has declined. If k is more expensive, the more it raises, the less can be spent on the other goods of the bundle. The issue is how to explore this result analytically.

One way of thinking to this problem is how to *compensate* the consumer for the increase of p_k by giving her some Δw so that her real income is the same as before. This allows us to isolate the effect of a shift in relative prices form the effect a change

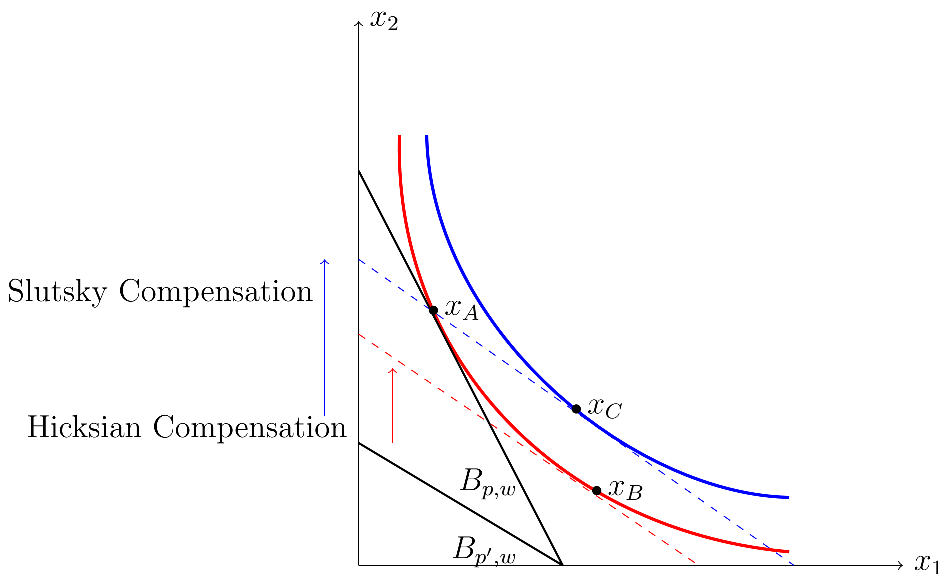

Figure 3: Slutsky and Hicksian Compensation

in prices has on the real income. The problem is now of determining how much Δw must be. There are two ways of answering this question:

- *Slutsky Compensation* is that Δw_S such that the consumer can buy back her original old optimal bundle x. This can be written as $w + \Delta w_s$. Note however, that the consumer can now choose a different bundle, since the old one can be no more optimal.

- *Hicksian Compensation* is that change in wealth, Δw_h, which allows the consumer to maintain her utility. Still note that, as seen before, the Hicksian Demand satisfies the Compensated Law of Demand, and therefore, as apparent in Figure 2, Hicksian Demand is a form of compensated demand.

These compensations are represented graphically in Figure 3. The consumer's original demand, at (p, w) is x_A. Then, the price of x_2 rises, so that the new budget set is $B(p', w)$. To compensate the consumer in order to stay on the original indifference curve, to her new real income must be added Δw_h (Hicks Compensation) so to reach the red dashed line (the Hicksian Compensation Budget Line). The new demand is x_B. In order instead of making the old demand x_A allowable, the consumer must be compensated with Δw_s (Slutsky Compensation) to reach the dashed blue line (the Slutsky Compensation Budget Line). Still note that x_A is not anymore an optimal bundle, so that the new demand is x_C.

This graph makes it apparent that a change in the price of x_2 affects the demand of x_1 in a way that involves both the Hicksian Compensantion and Slutsky. From the result above we can write the fundamental *Slutsky Equation*.

Proposition 7 (The Slutsky Equation (MWG 3.G.3)). *Suppose that $u(.)$ is a continuous utility function representing a L.N.S. and Strictly Convex $\succeq$ defined on $X = R^L_+$. Then, for all (p, w) and $u = v(p, w)$ we have:*

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \tag{16}$$

Proof. Recall that: $w \equiv e(p, v(p, w))$ and $h(p, u) \equiv x(p.e(p, u))$. Differentiating with respect to p_k we have:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,e(p,u)}{\partial p_k} + \frac{\partial x_l(p,e(p,u))}{\partial p_k}\frac{\partial e(p,u)}{\partial p_k}$$

Still, $\frac{\partial x_l(p,e(p,u))}{\partial p_k} \equiv \frac{\partial x_l(p,w)}{\partial w}$ and, by Shephard's Lemma, $\frac{\partial e(p,u)}{\partial p_k} \equiv h_k(p, u)$ and, finally $h_k(p, u) \equiv x_k(p, w)$.

Then we have the result:

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

□

The importance of the Slutsky Equation is that it decompose the demand change induced by a price change into two separate effects: the *Substitution Effect* and the *Income Effect*.

$$\underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution Effect}} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{\text{Total Effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Income Effect}}$$

Furthermore (16) can be arranged in a more economic meaningful way as follows:

$$\underbrace{\frac{\partial x_l(p,w)}{\partial p_k}}_{\text{Total Effect}} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution Effect}} - \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Income Effect}}$$

The economic intuition behind the *Slutsky Equation* is that if the price of good k increases, this has two effects on the demand for good l. The *Substitution Effect*, that is a movement along the original indifference curve, since the utility is fixed (i.e. Hicksian Demand refers to a EMP). And an *Income Effect*, that is the movement from one indifference curve to another. A change in prices determines a change in income, and therefore in the size of the budget line, which represents the constraint of the UMP.

These effects are represented graphically in Figure 4. A consumer faces an initial price-wealth situation (p, w) and therefore a budget set $B_{p,w}$.Then he chooses x_A. Let's assume now a change in the price of x_1, so then the new budget set is $B_{p',w}$. The new optimal consumption is x_C. But this move from x_A to x_C can be decomposed in two different parts. The *Substitution Effect*, which affects the Hicksian Demand. Since, by definition, $h(p,u)$ solve the EMP constrained to $u(x)$, the new demand must be on the same indifference curve. But since the lower price of x_1 makes it possible to reach an higher indifference curve, that is, new demand is x_C.

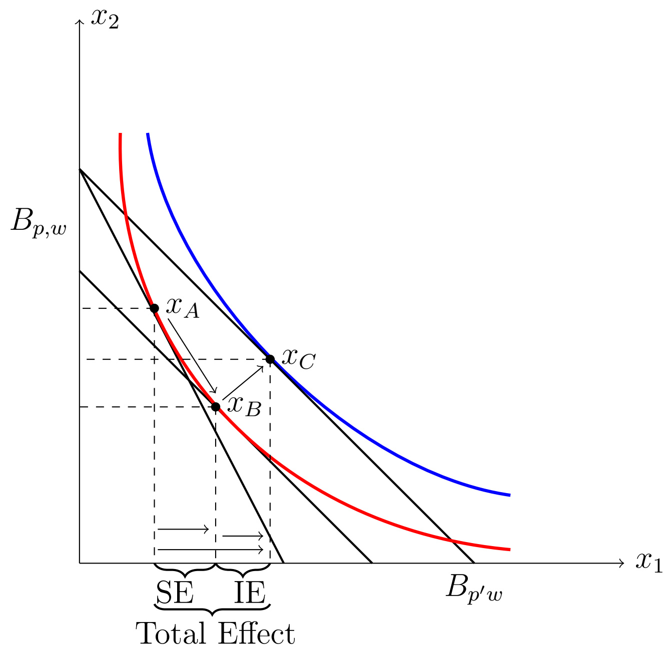

Figure 4: Total Effect, Substitution Effect and Income Effect

The importance of the *Slutsky Equation* is that it allows us to view comparative statics in prices as the sum of an income effect and a substitution effect. However, in order to fully assess this point one has to look at these effects in a more general way, that is, by rewriting equation 16 in matrix form. Indeed note that its right part is an element of the so-called *Slutsky Matrix* (see 15).

$$D_p h(p,u) = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} \dots & \frac{\partial h_1(p,u)}{\partial p_L} \\ \vdots & \vdots \\ \frac{\partial h_L(p,u)}{\partial p_1} \dots & \frac{\partial h_L(p,u)}{\partial p_L} \end{bmatrix} =$$
$$\begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \dots & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} x_L(p,w) \\ \vdots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \dots & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \end{bmatrix} = (S(p,w) \tag{17}$$

The advantage of this matrix form is that it shows the own-price substitution effects as well as the cross-price substitution effects. $D_p h(p,u)$ is the matrix of price effects for the Hicksian Demand, which is, roughly speaking, the equivalent, for the Hicksian Demand of the matrix of the Price Effects for the Walrasian Demand. However there are also important differences. Since we know that we can recover $h_i(p,u)$ simply differentiating the expenditure function $e(p,u)$, this means that if we are at an optimum in the EMP, the changes in demand caused by price changes do not affect the consumer's expenditure. Furthermore, if $h(p,u)$ is continuously differentiable at (p,u), we can state this important result concerning the properties of $D_p h(p,u)$.

Proposition 8. *Suppose that $u(.)$ is a continuous utility function representing L.N.S. and strictly convex preferecnces relation $\succeq$ defined on $X = R^L_+$. Suppose also that $h(p,u)$ is continuosly differentiable at (p,u). Then:*

1. $D_p h(p,u) = D^2_p e(p,u)$
2. $D_p h(p,u)$ *is a Negative Semidefinite Matrix*
3. $D_p h(p,u)$ *is a symmetric matrix*
4. $D_p h(p,u) p = 0$

Proof. 1. This follows from Shephard's Lemma by differentiation

2. This properties, as well as 3 derive from the fact that since $e(p,u)$ is continuous and concave, its Hessian Matrix is symmetric (by the properties of the Hessian Matrices) and Negative Semidefinite (since the function is concave).

3. This derives from the fact that $h(p,u)$ is Homogenous of Degree Zero in p. Then $h(\alpha p,u) = h(p,u)$, and therefore $h(\alpha p,u) - h(p,u) = 0$. Taking the derivative with respect to α we have $\frac{\partial h(\alpha p,u)}{\partial \alpha} p = 0$.

□

The economic meaning of Negative Semidefiniteness of $D_p h(p,u)$ is that if the price of i rises, then the change in $h_i(p,u)$ is not positive (i.e. $\frac{\partial h_i(p,u)}{\partial p_i} \leq 0$). This is a differential form of the law of the demand.[2] The symmetry of $D_p h(p,u)$ is a direct consequence of its being an Hessian Matrix, but its economic meaning is somewhat "fuzzy". Indeed, it means that the effect of a small increase of the price of good i on the quantity demanded of good j is identical to the effect of a small change of the price of j on the quantity demanded of i.

We can present an example of comparative statics (taken from: Kreps, 1990, p. 61). Let's first recap the different typologies of goods, with respect to wealth and price effects:

- Good j is said to be *inferior* if its wealth-derivative is less than zero: $\frac{\partial x_j}{\partial w} < 0$

[2] Because in the Hessian Matrix in the main diagonal we find all the second-order derivatives, and in the Negative Semidefinite Matrix these elements are always non-positive

- It is *normal* if its wealth-derivative is greater and equal than zero: $\frac{\partial x_j}{\partial w} \geq 0$.
- It is *ordinary* if its price-derivative is less than zero: $\frac{\partial x_j}{\partial p_j} < 0$.
- It is *Giffen* if its price-derivative is greater than zero: $\frac{\partial x_j}{\partial p_j} > 0$.

Let's consider the case of a *Giffen* good. Since a good is Giffen with respect to its own price-change, we have to look for those value across the main diagonal in all the matrices involved. So, in Slutsky Equation's terms:

$$\frac{\partial x_j(p,w)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_j} - \frac{\partial x_j}{\partial w} \cdot x_j(p,w)$$

$\frac{\partial x_j}{\partial p_j}$ must be greater than zero. But we know that $\frac{\partial h_j}{\partial p_j}$ is non positive, since it is an element of the main diagonal of an Hessian Negative Semi-Definite Matrix. Therefore, the only possibility for j's own price-derivative of being > 0 is that j is also inferior, i.e. that $\frac{\partial x_j}{\partial w} < 0$. But this is not sufficient. j must be sufficiently inferior so that its income effect overcomes the substitution effect. This could be the case if j occupies a great share in the consumer's consumption bundle.

Furthermore, from this matrix it is easy to identify different cross-price effects. Indeed it is sufficient to look at the sign of cross-derivatives. Then two goods, l, k are *substitutes* at (p,u) if $\frac{\partial h_l}{\partial p_k} \geq 0$; and *complementary* if $\frac{\partial h_l}{\partial p_k} \leq 0$. Finally, since $D_p h(p,u)$ is Negative Definite, and therefore $\frac{\partial h_i}{\partial p_i} \leq 0$, property 4 of Proposition 8 ensures that there must be a good k for which $\frac{\partial h_l(u,p)}{\partial p_k} \geq 0$, i.e. every good has at least one substitute.

There is another interpretation of the *Slutsky Equation*. Indeed it describes the relationship between the slope of the Hicksian demand curve and the Walrasian demand curve at prices p. This relationship is represented in Figure 5, for the case of a Normal Good. This represents demand curve for good 1, holding all other prices fixed. Note that the two demands are equal when p_1 is p_1^0. Furthermore, in the figure, Walrasian demand and Hicksian demand refer to the same utility level, i.e. $h_1(p, v(p,w)) = x_1(p,w))$ (see p. 7 of these notes). From the figure it is apparent that the slope of the Walrasian demand curve is less negative than the slope of the Hicksian demand for that price. That is the Hicks demand curve is less responsive to price changes than is the Walrasian demand curve. At level p_1^0 there is no income compensation. When $p > p_1^0$ income compensation is positive, because the individual needs help to remain an the same utility level. Finally, at $p < p_0^1$, the income compensation is negative, to prevent an increase in utility from a lower price.

To understand this, let's see the Slutsky equation again. Recall that own-price derivative is negative by definition for the Hicksian demand. In order of the Walrasian demand having a lesser slope than the Hicksian, the income effect must be positive. Therefore, good 1 must be normal. In the case of inferior goods, the relationship is reversed: the Hicksian demand is less negatively steeper than than the Walrasian demand.

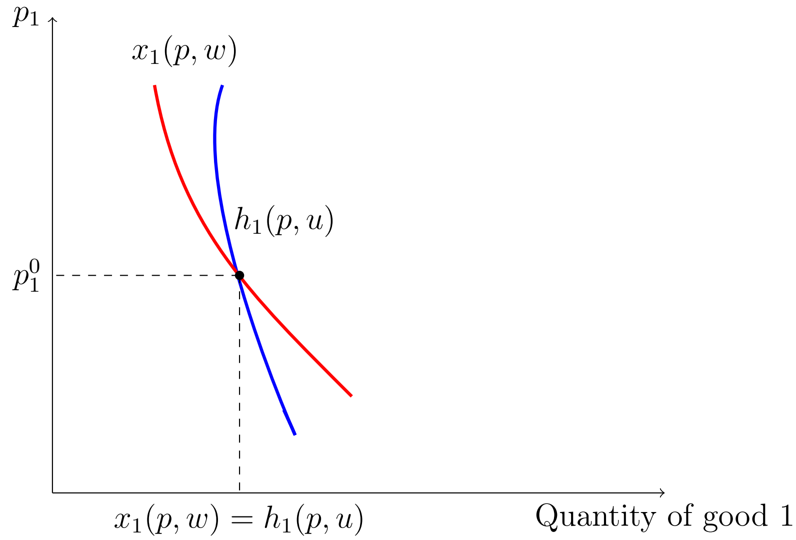

Figure 5: The Walrasian and the Hicksian demand for a Normal Good

5 Welfare Evaluation of price changes

Economists want to measure how consumers are affected by changes in prices and wealth. So far we have seen how consumers react to these changes. The issue is now to provide a way of measuring it. The simplest way of addressing this problem is by the notion of *Consumer Surplus* (see below). However this measure is mainly imprecise, and only in specific circumstances (addressed below) it can be considered exact.

The first problem to deal with is that we cannot really measure how utility changes as effect of some policy. To simplify, we consider a consumer with rational, continuous and Locally Non-Satiated $\succeq$, and furthermore, that both $e(p, u)$ and $v(p, w)$ are differentiable. Besides, the only focus will be on a price change, so that the wealth is fixed, and it is evaluated the impact of a welfare change from p^0 to p^1.

Let take (p^0, w) and (p^1, w), that is the pair representing the original prices and wealth, and the pair representing new prices and the same wealth. A simple way of seeing it is to compute the variation in the consumer's indirect utility:

$$v(p^1, w) - v(p^0, w)$$

Above there is, intuitively, the welfare change. If the difference between utility at new prices and old wealth and old prices and old wealth is positive, then, we could presume, the consumer has benefitted from this change.

However, we don't know what utility is, and the way we constructed utility functions aimed only to make ordinal utility representable. A possible getaway from this point is that of linking utility to money, using what is usually referred to as *Money Metric* Indirect Utility, which is constructed starting from the $e(p, u)$, and has the same properties (see Proposition 2).

Thus, choosing a price vector $\bar{p} >> 0$, and an indirect utility function $v(p, w)$, we can write the following:

$$e(\bar{p}, v(p, w))$$

Therefore, we can write the utility difference as follows:

$$e(\bar{p}, v(p^1, w)) - e(\bar{p}, v(p^0, w))$$

This function gives how much money are needed, at prices $\bar{p}$ to reach utility $v(p, w)$. In other words, this function measure how much income the consumer would need, at prices $\bar{p}$ to be as well off as she would be facing prices p and income w.

Assume that we are facing a change of prices from p^0 to p^1 (so that $\bar{p}$ can be either the new prices or the old one). Then, the question is: what is the impact on a given consumer, with an income w, of the change of p^0 to p^1?

Two measures of compensation can be employed. These are the *Compensating Variation* and the *Equivalent Variation*. We can define both of them in terms of $e(p, u)$ and $v(p, w)$. Recall that $v(p, w) \equiv u$.

So then, we can write CV as follows:

$$CV(p^0, p^1, w) = e(p^0, u^0) - e(p^1, u^0) = w - e(p^1, u^0) \tag{18}$$

Since $e(p^0, u^0) = e(p^1, u^1) = w$ (and $v(p^1, w) = u^1$ and $v(p^2, w)$, if the prices change from p^0 to p^1, CV tells how much we will have to compensate, or charge, the consumer to stay on the same indifference curve. It uses the new prices as the base.

Equivalently, EV can be written as:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w \tag{19}$$

That is, the change in expenditure that would be required at the original prices to have the same effect on consumer that price change had. In other words, it uses the current prices as the base and asks what income change at the current prices would be equivalent to the proposed change in terms of its impact on utility.

These variations are depicted in Figure 6. To each indifferent curve is associated a level of utility. Therefore, each budget set represents those combinations (p, w) through which the consumer obtains utility u and u^1. Assume that the price of x_1 decreases from p^0 to p^1. Now the consumer can reach a new indifference curve, therefore he can obtain higher utility. EV represents how much the consumer must be compensated in order to be as well off as when facing p^1. CV instead represents how much money should be taken away from the consumer in order to make her stay as well off as when facing p^0.

Recall that the classic tool for measuring welfare changes is *Consumer's surplus*:

$$CS = \int_{p^0}^{p^1} x(t)dt$$

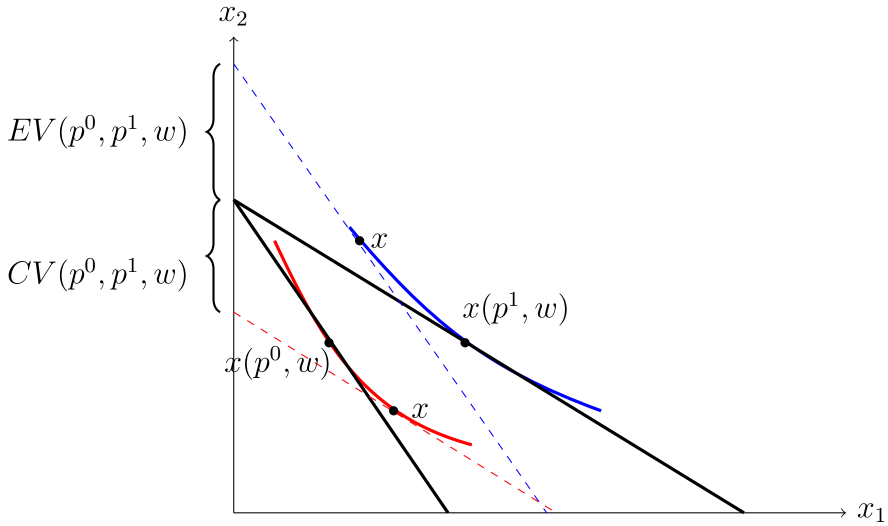

Figure 6: The Equivalent Variation and the Compensating Variation

Assuming a demand function $x(p)$, the Consumer's surplus associated with a price movement from p^0 to p^1 is the area to the left of the demand curve between p^0 and p^1. However, with one exception (if preferences are quasi-linear), usually, CS is not a precise measure of welfare changes, because $EV \neq CV$.

The EV and CV can be represented in terms of the Hicksian Demand Curve. Assuming that only the price of good 1 changes from p^0 to p^1, and $w = w^1 = w^0$, we can write:

$$EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_y(p_1, \bar{p}_{-1}, u^1) dp_1 \tag{20}$$

Where $\bar{p}_{-1} = (p_2, \ldots, p_L)$. Thus the change in Consumer Welfare measured by the equivalent variation is represented by the area between p^0, p^1 and the left of the Hicksian Demand for good 1 associated with utility level u^1, that is the blue lines in Figure 6.

Similarly, the Compensating Variation can be written as:

$$CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_y(p_1, \bar{p}_{-1}, u^0) dp_1 \tag{21}$$

This is the area between p^0, p^1 and the Hicksian Demand for good 1 at utility u^0, that is the area of red lines in Figure 3.

In words, we can say that the Compensating Variation is the integral of the Hicksian Demand curve associated with the initial level of utility, and the Equivalent Variation is the integral of the Hicksian demand curve associated with the final level of utility.

Assuming, as done in Figure 6, that good 1 is normal, $EV > CV$. This relation reverses in the case of good 1 being inferior.

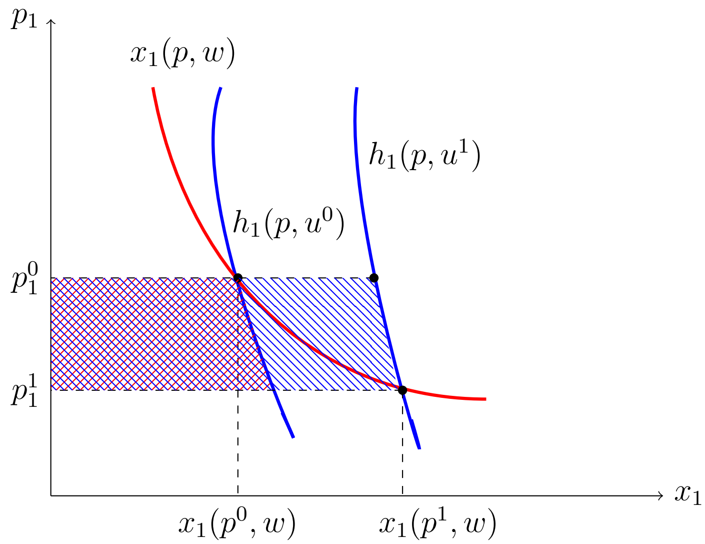

Figure 7: The Equivalent Variation and the Compensating Variation for a Normal Good

Furthermore, if preferences are quasi-linear (i.e. there is no wealth effect for good 1), $CV = EV$. In this last case (and only in this one), $CV = EV$ and correspond to the Consumer Surplus. In all other cases, this can be seen as no more than an approximation between Compensating Variation and Equivalent Variation.

Appendix I: How to derive the Walrasian Demand of a Cobb-Douglas Utility Function

Since the Walrasian Demand is the (uncompensated) optimal bundle which solves the UMP, we must solve the following:

$$\max_{x_1,x_2} A \cdot x_1^{\alpha} x_2^{1-\alpha} \text{ s.t. } \quad p_1 x_1 + p_2 x_2 \leq w$$

Note that for simplicity we assume only two variables (but the number can be larger) and $\alpha + 1 - \alpha = 1$ (the latter is an usual requirement of the Cobb-Douglas function: in the case where $x_1, \ldots, x_n$, then: $\sum_{i=1}^{n} \alpha_i = 1$).

Write the Lagrangian:

$$\mathfrak{L}(x_1, x_2, \lambda) = A \cdot x_1^{\alpha} x_2^{1-\alpha} + \lambda \cdot [w - p_1 x_1 + p_2 x_2]$$

Note that we can solve this by using the Kuhn-Tucker Method. Therefore, we write down the F.O.C.'s and the Complementary Slackness Condition:

- $\frac{\partial \mathfrak{L}}{\partial x_i} = 0 \quad$ for $i = 1, 2$
- $\lambda \cdot \frac{\partial \mathfrak{L}}{\partial \lambda} = 0$

Write down the F.O.C's:

$$\frac{\partial \mathfrak{L}}{\partial x_1} = \alpha A \cdot x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0 \tag{22}$$

$$\frac{\partial \mathfrak{L}}{\partial x_2} = (1-\alpha) A \cdot x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$$

And the Slackness Condition:

$$\lambda \cdot \frac{\partial \mathfrak{L}}{\partial \lambda} = \lambda \cdot [w - p_1 x_1 - p_2 x_2 = 0]$$

It is apparent that Slackness is satisfied if: $\lambda = 0$ or $\lambda \geq 0$. If λ is equal to zero the problem simply becomes that of unconstrained optimization of the Lagrangian (the constraint is multiplied by zero, and therefore it disappears). Let's explore the case when $\lambda \geq 0$. In this case, the Slackness condition is satisfied if and only if $w - p_1 x_1 + p_2 x_2 = 0$, and then:

$$w = p_1 x_1 + p_2 x_2$$

Note that this above is the Walras' Law. Solving the F.O.C's for λ we obtain:

$$\lambda = \frac{\alpha A \cdot x_1^{\alpha-1} x_2^{1-\alpha}}{p_1}$$

$$\lambda = \frac{(1-\alpha) A \cdot x_1^{\alpha} x_2^{-\alpha}}{p_2}$$

Equating the equations above, and since $\alpha - 1 < 0$, we can write:

$$\frac{\alpha A x_1^{\alpha-1} x_2^{1-\alpha}}{x_1^{\alpha-1} \cdot p_1} = \frac{(1-\alpha) A \cdot x_1^{\alpha}}{x_2^{\alpha} \cdot p_2}$$

Cross-multiplying, we obtain:

$$\alpha A x_2 p_2 = (1-\alpha) A x_1 p_1$$

Solving for x_1:

$$x_1 = \frac{\alpha A x_2 P_2}{(1-\alpha) A p_1} = \frac{\alpha x_2 p_2}{(1-\alpha) p_1}$$

Plugging x_1 in the constraint, we obtain:

$$p_1 \frac{\alpha A x_2 P_2}{(1-\alpha) A p_1} + p_2 x_2 = w$$

Simplifying p_1:

$$\frac{\alpha x_2 p_2}{(1-\alpha)p_1} + p_2 x_2 = w$$

Multiplying both sides by $(1-\alpha)$:

$$\alpha x_2 p_2 + (1-\alpha)p_2 x_2 = w(1-\alpha)$$

Finally, we have:

$$\alpha x_2 p_2 + p_2 x_2 - \alpha p_2 x_2 = w(1-\alpha)$$
$$x_2 = \frac{w(1-\alpha)}{p_2}$$

To find x_1 just plug x_2 into the constraint and solve:

$$w = \quad p_1 x_1 + p_2 \frac{w(1-\alpha)}{p_2}$$
$$p_1 x_1 + w - \alpha w - w = 0$$
$$x_1 = \frac{\alpha w}{p_1}$$

So, then, the Walrasian Demand of the Cobb-Douglas Utility Function is $x(p,w) = (\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_1})$.

This can be generalized, in the case of n-variables, as follows: $x(p_1 \ldots p_n, w) = ((\frac{\alpha_1 w}{p_1}, \ldots \frac{\alpha_n w}{p_n})$