Marginal Revolution, Economic Equilibrium and Mathematical Economics. Notes on Walras, Jevond and Fisher

Gianluca Damiani*

1 Origins and meaning of Marginal Revolution in Political Economy

Starting from the 1870s, economic theory went through dramatic changes that deeply affected the state of the existing theory. Obviously, this shift, namely the passage from labor-value theory to the so-called marginal utility theory, has been intended as a decisive step in the history of modern economics. To cover the widespread acceptance by economists of two fundamental concepts, marginal utility, and marginal productivity, John A. Hobson coined the word "marginalism" in 1914. (Howey 1972). The adjective "marginal" relates to the fundamental feature of the new theory, that is, the "substitution" at the margin between two different goods to which is attributed the same value by the individuals involved in the exchange. These values are computed through the employment of differential calculus.

The development of these theories in the last quarter of the XIXth century is of extreme importance for historians as well as for the scholars committed to investigating the scientific foundations of economics because it directly points out the problem of the theoretical and philosophical status of economic science. Moreover, the term employed by historians and economists to define this intellectual process is itself problematic. In fact, the "marginal revolution" refers to a process of scientific revolution, where the stress is put on the passage from the erroneous theories of the past toward the "scientific" present. The real nature of this revolutionary process is highly debated, notwithstanding his acceptance within the economics community is a matter of fact. (Collison Black, Coats, and Goodwin 1973)

In these notes, I want to address some issues of the history of the development of marginalism, focusing mainly on a short review of some alternative interpretations and a (brief) history and description of the employ-

*Ph.D. student, Department of Economics, University of North Carolina at Chapel Hill. e-mail: gdamiani@ad.unc.edu

ment of mathematical techniques between the 1870s and 1890s. What will be missing is a systematic reconstruction of the historiographic and philosophical issues concerning the genesis of marginal utility theory. Some remarks on specific topics and problems will be contained in the course of the text; otherwise, the obvious reference for a detailed account is Mark Blaug's 1997 pivotal volume. (Blaug 1997, 277 at ss.) What is noteworthy is that, according to Blaug, all the explanations of the genesis of marginalism are flawed, and consequently, there is not a "canonical" or "orthodox" interpretation.

The so-called "marginal revolution" has been associated with the theoretical works of three different authors who, each independently from the other, published three main treatises in the early 1870s to reject the "orthodox" economic theory. These, famously, were the English economist and philosopher of science William Stanley Jevons (1835-1882), the Austrian Carl Menger (1840-1921), and the French Leon Walras (1834-1910). Their works were, respectively, *the Theory of Political Economy*, 1871, *the Principles of Economics*, 1871 (or. *Grundsätze der Volkswirtschaftslehre*), and the *Elements of Pure Economics*, 1874 (or. *Elements d'Economie Politique Pure*). (Jevons 1879; Menger 1976; Walras 2014)

These three authors are often referred to as the founders, or the "first generation", of marginalist economists and were followed by many direct and indirect pupils, most notably Vilfredo Pareto, Francis Y. Edgeworth, Alfred Marshall, Friedrich von Wieser, Eugen von Böhm-Bawerk, Irving Fisher and Joseph Schumpeter. Around the first decade of the XXth century, in Great Britain, as well as in Austria-Hungary, the United States, and even in Italy, the marginalist approach, now defined as neoclassical economics, has become dominant, although alternative theories, namely historical schools of economics and institutionalism were not totally dismissed, at least until the Second World War. (Tribe 2003)

In France, the success of marginalism was less outstanding (although some French writers, like Augustine Cournot, were the forerunners of the employment of mathematics in economic analysis), but only in Germany, the whole approach was quite totally rejected until the 1920s.[1] Then, the most important and original debates surrounding economic theories were all treated within the marginalist framework. Moreover, around the 1900s, a considerable semantic turn occurred in the field, namely the replacement of "political economy", in the English-speaking world, with the more scientific-sounding "economics". This substitution has been first adopted by Jevons in the second edition (1879) of his work and has become the standard definition of the discipline from Marshall's *Principle of Economics*.

[1] As a consequence of the "methodenstreit", the debate, in german-speaking world, surrounding the methods in political economy, between Carl Menger, a proponent of economic theory, and Gustav Schmoller, one of the most devoted supporters of historical methods. (Menger 1996)

(Jevons 1879, pp. xiii–iv; Marshall 2009) However, "political economy" never disappeared, and still today is employed, although to define radical different sub-fields of the discipline.

Finally, this period also saw the definitive institutionalization of the discipline, mainly through the creation of new faculties and departments, starting from Cambridge University, where, in 1903, Marshall established the first undergraduate course in Economics, but also through reviews and associations (for instance, the American Economic Association was formed in 1885, the *Quarterly Journal of Economics*, Harvard's most prestigious economic review, started the following year, the Royal Economic Society in 1890).

One of the most evident hallmarks of this new analytical approach to economics was the revival of the concept of utility. Indeed, despite the fact that this concept has been widely used in social theories, political philosophy, and political economy well before the late decades of the XIXth century, only this period saw the decisive commitment of economists to the real problems of utility measurement, starting, with the problem of the measurement of pleasure for economic agents. So, the classical problem of economic value, that is, needing a fixed quantity of a given commodity (for example, labor) to determine prices, can be solved by applying a new way of treating utility theory. In this sense, the labor theory of value, the weakest part of classical political economy, could be replaced. The central role played by the notion of utility in economic theory is still maintained today, albeit with numerous transformations and deprived of all substantive content.

The marginal revolution involved a radical departure from the main lines of classical economics. (Hutchison 1978; Blaug 1997) For such authors as Adam Smith, David Ricardo, and John Stuart Mill, the economic system was determined by three distinct factors of production, land, labor, and capital. The land was not "augmentable", i.e., it was in fixed quantity, conversely to capital and labor. Then the function of economic analysis was to expose the effects of changes in the quantity and quality of labor on the rate of growth of aggregate output. This was a function of the rate of profit on capital, so secular trends in factor prices and in distributive shares arose naturally. The prices of products were obtained from the natural rates of reward of the three factors of production. Land rent was treated as the differential surplus of the cost of cultivating the marginal land. Wages were derived from the long-run costs of means of subsistence. Finally, the profit was a residual of prices minus costs. Letting apart rent theory, no considerations about the scarcity of the supply of factors of production were made. And despite a generic utilitarian framework, there was no room for a consumer's theory. But, the main problems of this theory concerned the determination of prices through a theory of value. In fact, it is often assumed that classical economists treated prices exclusively in terms of supply, namely in terms of production costs. This is the labor theory of value, i.e., the value of a good is determined by the quantity of labor used to produce it. In reality, as noted by some modern scholars like Blaug, there were at least two theories of value. Indeed supply determined only the price of industrial goods, while the price of agricultural goods varied with the scale of output and hence the pattern of demand. From this derives a "fatal indeterminacy" in classical distribution theory: ≪[...] since wage goods consisted largely of the products of agriculture, real wages depended on the position of the "margin of cultivation" and hence on the length to which investment was carried in agriculture.≫ (Blaug 1997, p. 281) This generated some issues regarding the real determinants of the three factors of production's distribution shares in the long run. The attempts to solve these puzzles occupied a vast part in the debates concerning political economy in the central decades of the XIXth century. In 1860s Mill's implicit abandonment of the classical value theory, as well as his recantation of the wages fund doctrine, opened the gates to new approaches, starting from the works of Fleeming Jenkin (the first British author to draw supply and demand curves) and Jevons himself. In particular, the latter wrote and read a rejection of the classical theory of value (with a first outline of marginal utility theory) in 1862. (Jevons 1866)

Parallel to these developments, in the 1850s and 1860s, there was also a revival of interest in Bentham (who died in 1832) and began the exploration of psychological attitudes toward the variation of sensations (in the works of Richard Jenning, from which Jevons will develop his "Law of Marginal Decreasing Utility"). Moreover, outside England, the labor theory of value had not been really accepted by the vast majority of economists, and many economic analyses were still conducted in a utilitarian fashion. After the 1870s instead, the essence of economic problems became the investigations of the conditions under which given productive services were allocated with optimal results among competitive uses. Optimality was intended in terms of consumer satisfaction. ≪For the first time, economics truly became the science that studies the relationship between given ends and given scarce means that have alternative uses for the achievements of those ends.≫ (Blaug 1997, p. 278) This transformation was made possible through the development of a theory of value grounded on what was immediately perceived as a unifying principle, i.e., the concept of marginal utility. This consisted of a generalization of the Ricardian rent theory. According to the latter, agricultural prices are determined by production under "the least favorable circumstances", namely, on marginal land. Ricardo applied this model only to land, which was fixed in quantity, and, consequently, not reproducible.

From the 1870s onward, economists began to apply this principle to all goods, combining it with utility analysis. The famous Adam Smith's paradox concerning water and diamonds[2] was solved because, in contrast to Smith and other classical economists, utility is not fixed in its esteem made by an economic agent. In fact, utility is decreasing with respect to the satisfaction of needs. This was what has been called the "Law of diminishing marginal utility". All we know about utility is the relative significance of an increment of one commodity with respect to a decrement of another. Marginal utility is decreasing because the strength of the response to a stimulus diminishes with each repetition of that stimulus in some specified time period (the Weber-Fechner Law). (Edgeworth 1881) Within this theoretical framework, the exchange happens because two individuals assign much marginal value to the commodity received over the commodity given. This doesn't mean that one commodity is more useful than another in every circumstance, but only that its relative marginal significance exceeds the other. This marginal significance is not fixed but varies for different persons and under different circumstances. Moreover, the same principles can be applied to production theory. What matters in production theory is the possibility, through combining different factors of production in different quantities, to obtain new products. As well as in utility theory, also in production theory, the marginal value, i.e., the quantity of product obtained by adding a single unit of factor, is decreasing. Exactly like in consumer theory, also in production theory, there is a point after which it is not convenient to employ another unit of a factor of production. Finally, the retribution of factors depends on their relative scarcity to consumers' wants for the products to be produced. This leads directly to the problem of "discovering" where these points are, namely how to efficiently allocate all the factors of production (for a firm) or all the goods (for a single consumer) to maximize satisfaction. An "efficient" allocation implies that each unit of a dividend is divided in such a way that the gain of transferring it to one use will be equal to the loss in withdrawing it from another (the "equi-marginal principle"). As stated by Blaug, ≪the whole of neo-classical economics is nothing more than the spelling out of this principle in even wider contexts, coupled with the demonstration that perfect competition does under certain conditions produce equi-marginal allocations of expenditures and resources.≫ (Blaug 1997, p. 280)

These are, very briefly, the main departure points of marginalist analysis with respect to classical political economy. All the theoretical debates in economic analysis, up to the 1930s and the mathematical revolution in the 1950s, concerned these issues, for example, how to measure utility more precisely, how to explain maximizing behavior of the firms, or how to expand exchange theory. But, the apparent "unity" of this analyt-

[2] That is, water is more useful than diamonds, but is essentially worthless; diamonds are essentially useless but their price is exceedingly high

ical framework cannot obscure the fact that there were some differences between the different schools and traditions, not only with regard to different techniques of analysis (the employment of mathematical tools) but also with regard to the precise nature of theoretical concepts like "utility". For instance, the emphasis on the allocation of given means with maximum effect is much stronger in the Lausanne and Austrian traditions than in the English School. All devoted their attention to the action of the competition in the allocation of resources in an essentially static framework through the concept of general economic equilibrium. But even in this latter case, the way equilibrium was intended and used varied exceedingly.

2 Early debates in Mathematical Economics: Jevons and Walras

Despite mathematical economics becoming predominant in the discipline only after the 1950s, the number of works, theses, and papers devoted to defending this approach augmented sharply in the decades following Walras', Jevons, and Edgeworth's main works. In the early 1900s, a young French scholar committed to mathematical methods, Jacques Moret, summarized the main objections against the employment of mathematics in economics along two different lines. (Moret 1915) One kind of objection came from mathematicians, given their skepticism toward social sciences and also applied methods outside more traditional fields, namely physics and engineering. The other and more important objection came from the economists' community itself. Three types of criticism are listed by Moret: the first concerns the presumed sterility of mathematical methods; the second is about the difficulties of mathematical studies for an economic curriculum; finally, the third addresses the foundations of economics, namely being a moral science and not a mechanical discipline. It is clear that this list comprises some of the objections continuously levied against mathematical social sciences today, mainly the criticism of mathematical "reductionism" to treat the complex phenomena of the real world.

Mathematical economists' responses to these objections were different well before Moret's study. Two were the main arguments used to defend the utility of mathematics in economic theory. One was that adopted firstly by Augustine Cournot, followed explicitly or implicitly by Jevons and Walras in their main works, as well as in some specific works devoted to the argument. Here the stress is put mainly on the use of arbitrary functions (i.e., functions that are asked to merely satisfy certain restrictions) to describe economic processes instead of a computational approach. Moreover, economics is defined as a discipline that is intrinsically mathematical due to its treatment of quantitative relationships.

Cournot, who was a well-trained mathematician (he was one of the pupils of Simeon-Denis Poisson), defended the links between mathematics and economics at two different levels. More philosophically, in his *Principes de la Theorie de la Richesse* (1863), an exclusively verbal exposition of his economic theory, where Political Economy is seen as opposite to Law and Jurisprudence because these are related to individual cases, the earlier to "great numbers". Instead, in his most famous and important *Recherchés sur les principes mathématiques de la théorie des richesses* (1838), he explains the use of mathematics in economics on the basis that this discipline is deeply rooted in the ideas of numbers and measurement. In particular, his scope is not to find numerical results but instead to ascertain what form of relation exists between two or more economic quantities. Then, that branch of analysis that comprises arbitrary functions can be employed, and the theory of function and the first principles of differential and integral calculus could be sufficient. For example, the relation between quantity demand and price may be presented in simple, functional terms as $D = F(p)$ and can be sufficient to know some of its properties, i.e., being decreasing and continuous. (Cournot 1927).

A different justification was adopted by Irving Fisher (also himself a well-trained mathematician), deeply influenced by the mathematical and physical approach of such scholars as the mathematicians Josiah W. Gibbs and Benjamin Peirce. Then, according to Fisher, the utility of mathematical methods in economics is strictly related to the main features of mathematical reasoning.

However, these are not stiff distinctions. In fact, as shown by many scholars, the theoretical analyses of marginalists, their differences apart, are profoundly embedded in the surrounding debates regarding measurement in psychology, physics, and mathematics. (Moscati 2018) So, even the scholars who employed arbitrary functions were committed to measurement problems. Fisher, instead, as it will be shown, fluctuated between a computational approach and a preference treatment of utility.

Not all the criticisms against mathematical economists came from the side of those authors and intellectual groups who radically rejected or opposed marginalism (for instance, the historical schools). Besides, the importance of mathematics has been recognized by some classical economists, although often only sketched. The most important example is the English polymath William Whewell (1794-1866), who went well beyond generic references to the implicit mathematical character of political economy, exposing instead Ricardian theory by means of equations.

By contrast, the group who accepted marginalism, repudiating at the same time the application of mathematics, comprises Carl Menger and his followers, from which originated the "Austrian School of Economics". Despite internal differences, the Austrian economists followed an apriori approach to economics, where no place is found for mathematical generalization. This has become more apparent after the radical mathematization of economics, but notwithstanding neither Menger nor Wieser, Eugen von Böhm-Bawerk, or Ludwig von Mises adopted mathematics, not even in simplest terms. The Austrian case is interesting because the rising prominence of mathematics came not through utility theory but instead through the concept of substitution at the margin, which, as seen, is the most important feature of marginalism.

But, Menger and Austrians apart, from the early XXth century, the majority of all great economic theorists employed at least an "intermediate level mathematics" involving mainly calculus, which was adopted because the aim of economic analysis was to maximize some economic function and elementary linear algebra because the economic system can be represented as a system of linear equations.

Different kinds of mathematical reasoning were present in economic theory well before the last quarter of the XIXth century. In fact, starting from Jevons' work, many authors filled the prefaces of their works with references to precursors as a way to legitimize their own innovative approach. Moreover, many works contained detailed discussions about the role and utility of mathematics in economics. This is the case of Jevons himself, Walras, Fisher, and others. Similar discussions can also be found in the pages of many important economic reviews. For instance, a bibliographical and chronological list has been compiled by Jevons in 1878 and published in the *Journal of the London Statistical Society*. (Jevons 1879, xxiii et ss) This list has been augmented by Fisher and attached as an appendix of the English translation of Cournot's work, edited by the American economist itself. (Cournot 1927)

If it is reviewed the first-hand literature concerning the mathematization of economics, it seems clear that the application of mathematics is strictly related to three main points. The first is the problem of consumer behavior, namely the satisfaction of pleasures and, more broadly, how to model individual behavior. This is the principle of Marginal Utility, the "heart", as yet seen, of marginal revolution. The second concerns the concept of equilibrium in an economic system. Finally, there are issues regarding economic policies to be adopted. Obviously, all these are nested. For instance, equilibrium theory, as a "research program", as convincingly shown by Ingrao and Israel, is deeply connected with the study of economic behavior as well as with the debates on free market policies. (Ingrao and Israel 1987) The generality of this latter point is more debatable, but at least up to the mathematical formalist revolution after the 1930s and the debates surrounding "socialist calculation", equilibrium theory used to be mainly associated with free-market economic policies.

Walras' 1876 essay offers a clear example of the intertwined treatment of all these points. This was published in 1876 in *Giornale degli Economisti*.

As showed by Piero Barucci, the Italian economists' community was partly receptive to new economic ideas, and this economic journal was deeply connected with the spreading of Marginalism in Italy (under the direction of such authors like Maffeo Pantaleoni and Antonio de Viti de Marco). (Barucci 1972) However, in the early years of its publication, Giornale's approach was more pluralistic, and much space was also devoted to methodological debates. An example is Gerolamo Boccardo's 1875 essay, where the Genoese author made some positive references to the application of mathematics in economics but without entering into details and focusing only on the use of mathematics as a quantitative method. (Boccardo 1875) Instead, Walras' essay, published the following year, is exclusively committed to defending mathematical economics.[3] This essay is divided into three parts. The first and the second entail an analysis of the main characteristics of economics and, overall, mathematical economics. In the third, the Lausanne professor confronts himself with the works of Jevons and Cournot.

Walras' starting point is a sense of dissatisfaction with the mainstream explanations of the beneficial effects of economic competition, as well as with the criticisms levied against this. According to the vast majority of economists contemporary to Walras, the benefits of the competition were largely positive, following the principles of "Laissez Faire, Laissez Passer". Albeit his own support to the latter, his aim is that of exploring further the features of an economic system through a detailed analysis of consumption and production theory. An analysis that, in his own view, was totally missing in the works of contemporary colleagues. Then, the issues at stake in pure economics involve the elaboration of a theory of exchange and a theory of production.

Moreover, he introduces a distinction between pure economics and applied economics. The first refers to the study of production and consumption in a competitive system from an exclusively theoretical point of view. Conversely, the second aims to describe if the effects of a competitive economic system are beneficial or not. Formally the problems of pure economics are the following: given different commodities and different quantities, a theory of exchange has to represent an economic system as a system of equations whose roots are the prices of the commodities. Instead, for a theory of production, the roots of equations encompass the quantities of products, their prices, and the prices of factors of production. Here we have, in verbal terms, a broad definition of economic equilibrium. To successfully employ mathematics in economic theory, Walras introduces a further distinction, namely "practical political economy". This entails all the issues regarding real economic activities, for instance, business. To put it briefly, thus, two different kinds of mathematics are needed. In the second

[3] This essay was written in French and translated into Italian by Boccardo. I was not able to find an English translation of this essay. see Walker 2006, 2006, ch. 8

case (that of applied political economy), the main problem is to compute exact quantities to foresee market operations. In the first case, instead, the role of mathematics is to explain and to model market equilibrium. Walras' example is the concept of effective demand, which is a decreasing function of price, that is $D = f(p)$, where $\frac{dD}{df(p)} < 0$. The same holds for utility, which is determined by "raretè" and is itself a decreasing function of quantity. These are arbitrary functions, but at the same time, the mathematical properties of these functions (namely their differentiability) make it possible to represent the conditions at which an exchange can happen and the conditions where supply equalizes demand. Then, to him, pure economics could be defined as a "new branch of mathematics". (Walker 2006, 102 et ss.) Jevons' theory of the final degree of utility, and Cournot's theory of demand, despite some weaknesses in Walras' own eyes, are two examples of the kind of pure economics made possible through the employment of mathematics.

To sum up, in this essay, Walras started from a practical situation, that is, the effects of economic competition, to show how this, as a problem of pure economics, can be better addressed with mathematical reasoning. Then, conditions of equilibrium, utility, and demand are discussed.

Walras' commitment to the quantitative nature of mathematics is important, albeit not explicit, as in Jevons' work. The English economist, in the preface of the second edition of his main work, contends that all economic writers must be mathematical because they treat economic quantities and the relations of such quantities. In this sense, it is impossible, according to him, to elaborate economic theory without any kind of mathematical reasoning. (Jevons 1879, xx, 3 et ss) Jevons' theory of economics, in his own words, is considered as being purely mathematical in character, and his main feature consists in applying differential calculus to such notions as wealth, utility value, demand, supply, capital, interest, labor, etc. Jevons was devoted to formulating a pure theory, treating economic laws in terms of functions, but also he was deeply convinced that these laws could be empirically demonstrated and measured. Indeed, even if ≪[...] a unit of pleasure or of pain is difficult even to conceive; [...] it is the amount of these feelings which is continually prompting us to buying and selling, borrowing and lending, laboring and resting, producing and consuming; and *it is from the quantitative effects of the feelings that we must estimate their comparative amounts.*≫ (Jevons 1879, p. 11, italics in the text). Then, he never attempts to estimate the whole pleasure gained by purchasing a commodity, but his theory states that a man, exchanging a good with another, derives equal pleasure from the possession of a small quantity of the good obtained with respect to that offered.

Jevons assigned great importance to the measurement of phenomena. Thus, the only way to advance scientific knowledge was the invention of suitable instruments of measurement, conceived as a ratio between the magnitude to be measured and a fixed unit. In fact, concerning economic theory, Jevons was strongly influenced by Jeremy Bentham's utilitarianism and his "calculus of pleasure and pain". To better delimit the scope of economic theory, the concept of utility must be shrunk, through a hierarchy of pleasures and pain, from "mere physical pleasure or pain" to honor or public shame, passing through "mental and moral feelings". The economic calculus is possible because pleasure and pain can be defined as "quantities", meaning that they can be "more or less in magnitude". But, Jevons recognized that pleasure or pain cannot be measured in the unit-based sense. (Moscati 2018, 28 et ss.)

Jevons' formal reasoning is introduced in chapter III of his work, where utility theory is discussed more properly. The main properties of this theory are derived "empirically" from the observations and discussions around human behavior. Then, satisfaction decreases with respect to quantity. Only after having drawn a two-dimensional cartesian system, where the intensity of utility is represented on the ordinate and quantity on the abscissa, and having plotted a decreasing line (curve) representing total utility, Jevons explicit his mathematical approach. (Jevons 1879, 46 et ss.) Indeed, if the total area below the plotted curve represents the total utility of a given good for a consumer, each point on this curve represents the utility of a single unit of good. A generic utility function, in the form of $u(x)$, where x is the quantity of good, that is, the independent variable, is written. Jevons does not explore any real mathematical property of his function (namely, he simply assumes this being continuous and differentiable, but without spelling it out), then differential calculus can be employed to compute the degree of utility for any quantity x. In other words, the degree of utility corresponds to the first derivative of the utility function. From an economist's point of view, what matters is the increment to total utility offered by the consumption of another unit of good. For this reason, Jevons employs the term "final degree of utility" to define ≪the degree of utility of the last addition, or the next possible addition of a very small, or infinitely small, quantity to the existing stock.≫ (Jevons 1879, p. 51) Needless to say, Jevons' final degree of utility corresponds to the marginal utility of a good. A given commodity must be distributed between different uses in an optimal way to obtain the equalization of the final degree of utility of the different uses. Here it is clear that, for him, economic behavior is a maximizing behavior under constraints, and the fundamental principle is that of equalizing marginal values. From a substantive perspective, the more important feature of employing differential calculus is the possibility to express value in terms of the ratio of exchange between final degrees of utilities.

Jevons' theory of exchange is grounded on the following principle: the ratio of exchange between any two commodities is the reciprocal of the ratio of the final degrees of utility of the quantities of commodity available for consumption after the exchange. If two commodities are defined as x and y, their ratio of exchange is the ratio between an infinitely small quantity of the first to the second which is given for it. In an exchange (which, in Jevons' example, involves two individuals and two commodities), the equilibrium point will be reached when an infinitely small amount of commodity exchanged in addition, at the same rate, will bring neither gain nor loss of utility. (Jevons 1879, 95 et ss.)[4]

The purpose of this paragraph has been to display Walras' and Jevons' attitudes toward mathematical economics briefly. Moreover, it is extremely difficult (and beyond the scope of these pages) to explore all the content of Walras' thoughts about mathematical modeling in social science (to explore further Walras' system of ideas: Walker 2006).

What deserves to be noted is that both authors adopted the concept of arbitrary functions to model economic behavior and describe economic systems. But Jevons, as noted by Walras, did not explore the mathematical (albeit elementary) properties of his functions (in contrast to Cournot and to Walras himself). At the same time, Jevons' stress on the quantitative content of economic issues is quite problematic. In fact, on the one hand, these originated economists' interest in measuring these quantities. On the other, the mathematical character of economic theory has been separated by statistical measurement and applications for more than a century from marginalists' early treatment. This means that it was not through quantitative analysis that mathematics became ever and ever more central in economics. This also means that the role of mathematics in economic theory must be justified in a quite different way from how Jevons, some of his followers, and Walras did. It seems to me that this was the path pursued by Irving Fisher, who was (perhaps with Cournot's exceptions) an ostensibly more gifted mathematician than any economist who preceded him.

3 Fisher's mathematical investigations

Irving Fisher (1867-1947) has been probably the most important American theoretical economist in the years between 1895 and 1940s. His fundamental contributions spanned from utility theory to monetary theory and theory of interest and capital, from index numbers to the early steps of econometrics. (Dimand 2019) Moreover, Fisher was deeply mathemat-

[4] If x and y are two commodities, respectively corn and beef, these are owned in different proportions by two agents, A and B in the following proportions: A holds $(a–x)$ corn and y of beef; B holds x of corn and $(b–y)$ beef. Then, we can represent the final degree of utility of corn for A as $f(a–x)$ and for B as $f(x)$; the same for the beef, the final degree of utility of beef for A is $g(y)$ and for B is $g(b–y)$. Then, the equilibrium point corresponds to the following equation: $\frac{f(a–x)}{g(y)} = \frac{y}{x} = \frac{f(x)}{g(b–y)}$. That is, the ratio between the final degree of utility of corn and beef for A must be equal to the ratio between the final degree of utility of corn and beef for B. (Jevons 1879, p. 100)

ically gifted, and expanded his math training during his undergraduate and graduate years at Yale University. Here, apart from William Graham Sumner, the famous (albeit not mathematically inclined) social scientist, his major intellectual influencer was the notable mathematician and physics Josiah Willard Gibbs (1839-1903) (for an overview of Fisher's life and career, see: Barber 2005). Then Fisher's treatment of utility theory, in his doctoral dissertation, published with the title *Mathematical Investigations in the Theory of Value and Prices* in 1892, is pursued in a mathematical fashion that resembles that of modern intermediate microeconomic courses. (Fisher 1892) Fisher, in fact, began his professional career as a tutor in mathematics at Yale before focusing exclusively on economics, and in 1897 he published a brief introduction to infinitesimal calculus (and co-authored an introduction to geometry) (Fisher 1897; Barber 2005)

Without further exploring his contributions to issues such as interest, capital, and empirical measurement, I want instead to focus on his mathematical treatment of utility and equilibrium analysis in his doctoral thesis.

Mathematically speaking, his equilibrium theory is a clear example of how the problem used to be addressed before the 1930s. But its importance also relies on Fisher's explicit handling of utility in terms of differential calculus (first and partial derivatives) and in his use of vector analysis to describe utility as a function of the quantities of two different goods. In this latter case, he draws indifference curves, yet employed by Edgeworth, to show the equality of marginal utility ratios between two goods. But contrary to the English economist, whose indifference curves served to illustrate cardinally measurable utility functions, Fisher's analysis depends on the concept of preference. (Dimand 2019, 23 et ss.) Moreover, he explicitly treats the space and properties of utility surfaces (utility curves) in terms of vectors.[5] Indeed, his text represented the first use of vector analysis in economic theory. (Mirowski 1992, p. 223)

Due to his employment of vector algebra, as well as of many physical metaphors (mechanical analogies, and furthermore, his famous description of economic equilibrium through the analogy with a hydraulic model), Fisher occupies a central role in Philip Mirowski's narrative concerning the development of neoclassical economics. In fact, for the latter, ≪[. . .] Fisher's thesis was the first (and last) published work [of neoclassical economics] to explore the physical metaphor in great detail. [. . .]≫ (Mirowski 1992, p. 223) Then, Mirowski has seen Fisher's work as the canonical neoclassical model, to argue, from a table of mechanical analogies Fisher put in his volume (Fisher 1892, pp. 85–6) that neoclassical economists were obsessed with rigid, inappropriate analogies to physics. Furthermore, these

[5] In fact, he defines the "maximum direction" of a commodity bundle (a combination of two different goods, A and B) that is how an individual can maximize its changing of the composition of the commodity bundle (more A and less B, or vice versa), as the "normal" vector of indifference line. (p. 74 et ss.)

analogies were out of date with modern physics and incorrectly understood by these authors. (Mirowski 1992, 223 et ss.) But this thesis (as well as many of Mirowski's analyses) has been widely criticized by other scholars, such as Robert Dimand. In fact, Fisher employed mainly examples taken from hydrostatics and not from mechanics, and these seem only a way to describe, in simple and necessarily incomplete terms, the functioning of a competitive equilibrium. According to Dimand, then the influence of Gibbs was crucial in a way quite different from the internalization of an implicit physics-like approach to the economic realm. Fisher's background in Gibbsian physics led him to construct a physical model to see how the equilibrium would look like. (Dimand 2019, 28 et ss.) Then, Fisher was the first author to explore the conditions of a competitive equilibrium, directly emphasizing his analogy with the real (i.e., physical) world. His training in applied mathematics stressed the importance of computability to display how equilibrium can be reached. Notwithstanding, he follows lines very similar to that of Walras, and like him, he believed that having the same numbers of unknowns and independent equations guaranteed the existence of a solution, overlooking non-negativity constraints on quantities.

More generally, his equilibrium analysis is based on modeling utility theory. This is defined in terms of "desires" rather than psychological attitudes or cardinally measurable quantities. Consequently, already in the first pages of his dissertation, the longstanding debates about the real significance of "utility" are dismissed, and the association between utility and pleasure is deferred to psychology. In his view, the economist's task was not to build a detailed psychological theory but instead to offer an explanation of economic facts. Economics does not deal with "pleasure" but instead with "desire", which is characterized in terms very similar to the modern concept of "preference". (Moscati 2018, p. 55) Notwithstanding, Fisher continues to employ the term "utility" in all his analyses.

Given two different quantities (numbers) A and B, three different situations are possible: A is equal to B; A is preferred to B; a unit of measure between A and B can be defined. This third point is historically important because it connects Fisher's analysis with the contemporary debates about utility measurement.[6]

[6]This point has been explored by some scholars, for example, in Moscati's work. (Moscati 2018, 55 et ss.) To sum up, this is based on the concept of marginal utility. In order to identify a unit of utility (that is, a unit of Marginal Utility), this must not depend on the quantities of other commodities (in current economic theory, it means that the utility function must be additive). Then, the utility can be measured as follows (this is Fisher's own numerical example): an individual consumes 100 loaves of bread and B gallons of oil per year. For this individual, the Marginal Utility of 100th loaf is equal to that of an increment β over B. Then, $U(100) = U(\beta)$. Fisher hypothesizes also that $U(150) = U(\frac{\beta}{2})$. If $U(\frac{\beta}{2}) = \frac{U(\beta)}{2}$, then $U(150)$ is $\frac{U(100)}{2}$. In this sense, $U(\frac{\beta}{2})$ could be used as the unit of marginal utility, and then $U(150) = 1$, $U(100) = 2$. (Fisher 1892, 14 et ss.) In other words, the marginal utility of any arbitrarily chosen commodity on the margin of some arbitrar-

In equilibrium analysis, utilities of A and B are represented as two different functions (continuous and differentiable), $U(A)$ and $U(B)$, i.e., the utility is a function of quantity. The marginal utility of A and B can be computed as the first derivative of the $U(A)$ and $U(B)$, that is, $\frac{dU}{dA}$ and $\frac{dU}{dB}$. If dA and dB are exchanged at the same ratio of A and B, we can write $\frac{A}{B} = \frac{dA}{dB}$, which, with some elementary algebraic manipulation becomes $(\frac{dU}{dB}) \cdot B = (\frac{dU}{dA}) \cdot A$, namely, the marginal utility of B times B (the quantity of B) is equal to the marginal utility of A times A. This is the general form that represents the value of a given good for each quantity of that good. The total utility can be represented as the integral of, respectively, $U(A)$ and $U(B)$, and the yield of exchange (or consumer's rent) as the difference between total utility and the utility value. (Fisher 1892, p. 18)

In the second part of the volume, utility is instead treated as a function of different commodities. This involves the concept of "complementary commodities" like, for example, butter and bread. The utility of butter for a consumer is determined by bread's quantity (and price) because these two goods are often consumed together. Mathematically, given two commodities, A and B, the utility function can be written as $U(A, B)$, and the marginal utility of A is the partial derivative of $U(A, B)$, namely $\frac{\partial U}{\partial A}$ (and, obviously, the marginal utility of B is $\frac{\partial U}{\partial B}$). (Fisher 1892, 64 et ss.)

Economic equilibrium is explored by Fisher through a method of increasing abstraction. In fact, he starts from the simplest case, namely when one good must be divided between many consumers (Fisher 1892, 26 et ss.) and when many goods must be divided between one consumer (pp. 31 et ss.) to show how an allocative equilibrium can be reached. Later he examines a situation involving m-commodities and n-consumers (pp. 35 et ss.), the combination between consumption and production (pp.54 et ss.), and finally, he tries to decompose the production process (pp. 60 et ss.). Although, as stated before, these pages are filled with many drawings of hydraulic models, he also offers an analytical self-contained mathematical treatment of the problem. All of his analysis is founded on some preliminary hypotheses (p. 25):

- a price-taking assumption for each consumer;
- the exchange occurred within a given period (one year);
- the equality of production and consumption rates;

ily chosen quantity can serve as a unit of utility, a concept that defines a "util". In this analysis, Fisher deliberately ignores the problem of complementarity and substitution between goods, recently addressed by two Austrian authors, Rudolf Auspitz (1837-1906) and Richard Lieben (1842-1919) in Austria. (Auspitz and Lieben 2015) In the second part of his volume, when Fisher removed the assumption of independency of the utility of each commodity from the quantities of the others, his method of measurement broke down. If the utility function is not additive, utilities cannot be measured on a ratio scale and can be ranked only according to the properties listed above.

- the utility of each consumer does not change;
- all the goods are infinitely divisible;
- marginal utility is decreasing with respect to the consumption (the same holds for disutility, i.e., for production);
- utility (as well as disutility) is a function only of the quantity of one good.

If the consumer is price taker, he consumes A up to $\frac{dU}{dA} = p$, i.e., the marginal utility of A is equal to p. The same hold for a producer, which produces until his marginal disutility is equal to the price of A. If a single commodity must be divided between different consumers, it will be divided in such a way that marginal utilities are equal for all the consumers. This also corresponds to the price of that commodity. Then, the equilibrium conditions can be written as follows:

- for each n-consumer, $\frac{dU}{dA_n} = F_n(A_n)$
- $A_1 + A_2 + A_3 + \cdots + A_n = K$
- $\frac{dU}{dA_1} = \frac{dU}{dA_2} = \cdots = \frac{dU}{dA_n}$

The first condition says that for each consumer, marginal utility measures the consumption of the commodity (in other words, he represents the utility function). The second condition indicates the total amount of that commodity in the market. Finally, the third condition is the equality of marginal utilities. Mathematically speaking, each condition represents a number of equations and unknowns. The two unknowns are the quantity of each commodity consumed by each individual and the marginal utility of that commodity. The first condition can be represented as a system of n-equation (each for any n-consumer) and $2n$ unknowns (each equation has two unknowns). The second condition is a single equation but without any new unknowns. Instead, in the third condition we have $(n–1)$ equations and no new unknowns. Here we have a clear example of the "classical" determination of economic equilibrium, well before the mathematical debates in the 1930s-1950s. In fact, without entering the details of the equations, Fisher states that the system is determined, i.e., we have a solution because the number of equations is equal to the number of unknowns. Actually, we have $n + 1 + (n–1) = 2n$ equations, and $2n + 0 + 0 = 2n$ unknowns.

The general case of equilibrium is that of m-goods and n-consumers (pp. 35 et ss.). Here, the previous implicit assumptions about the constancy of the marginal utility of money (in the first case) and the constancy of prices (in the second case) are relaxed, and so, for each consumer, the marginal utility of m-good is equal to the marginal utility of money times the price. Then, for each consumer, we have the same ratio between the marginal utility of different goods and their prices. Analytically, we have n-consumers, m-goods. The unknowns are the good consumed by each consumer and the quantity consumed of each good, the marginal utilities of each good and of each quantity consumed, and the prices of each quantity.

Consequently, the equilibrium conditions are the total amount of goods, which can be represented as a system of m-equations and $m \cdot n$ unknowns; the total amount of income, which is a system of n-equations and m unknowns (the prices); the utility functions for each consumer and for each quantity consumed (that is, $m \cdot n$ equations and $m \cdot n$ unknows, i.e., the marginal utilities); finally, the proportionality between marginal utilities and prices, that is, $n(m–1)$ equations. The system is determined because also, in this case, the number of equations corresponds to the number of unknowns.

Fisher's mathematical treatment of utility and economic equilibrium is not the only important part of his work. In fact, also his defense of the utility of mathematics in economics is extremely interesting and original. This is contained in the third (and last) appendix of his work (Fisher 1892, 105 et ss.). Differently from Jevons (although he reports a quote from the latter's preface of the 2nd edition of his 1871 work), he does not explicitly connect mathematical methods with the intrinsic quantitative nature of economics but prefers instead focusing on some features of mathematical reasoning. In fact, in his view, the scope of mathematics is that of judging the inner consistency of a theory and not of discovering new laws (a vision borrowed from the Harvard mathematician Benjamin Peirce). This essential point, according to him, is not understood by many critics of mathematical methods in economic and social theory. Therefore there is often a misreading of the mathematician's, as well as physicist's, work.

Some people, in his view, ≪[...] imagine that a physicist can sit in his study and with the calculus as a talisman spin out some law of physics. Some economists have hoped for a similar mysterious use of mathematics in their own science.≫ (Fisher 1892, p. 107) If mathematics does not have a constructive role, but it is a way of reasoning intrinsic to every science, we must distinguish between *mathematics* and the employment of *mathematical methods* (italics are in the text). The latter pertains only to determinate problems because it involves the utilization of mathematical symbols as well as mathematical operation (that is, rules to employ these symbols correctly, for instance, the rules of differential calculus). Then, the utility of mathematics depends on some specific circumstances, notably the capabilities of the scholar, the degree of mathematical theory used, and the degree of sophistication of the problem to solve:

> ≪The formulae, diagrams, and model are the instrument of higher study. The trained mathematician uses them to clarify and extend his

previous un-symbolic knowledge [...] to think of velocity, acceleration, force, as fluxions is not to abandon but to supplement the old notions and to think of momentum, work, energy as integrals is greatly to extend them. Yet he is well aware or ought to be that to load all this on the beginner is to impede his progress and produce disgust. So also the beginner in economics might be mystified, while the advanced student is enlightened by the mathematical method.≫ (Fisher 1892, p. 108)

As well as for the development of modern physics, mathematics in economics is useful as the latter becomes more complicated. But, at the same time, it has helped to correct the mistakes of previous theories through the discovery of the concept of marginal utility. And although the earlier stages of this theory are, in reality, the adjustments of older theories, mathematical economics has also been capable of offering new original contributions. Among the few cited from Fisher, we found the concept of consumer rent and the equilibrium conditions for producers and consumers. (Fisher 1892, 111 et ss.) Fisher's argument can be summed up by the following citation, with which he purposely closed his appendix:

≪The effort of the economist is to see, to picture the interplay of the economic elements. The more clearly cut these elements appear in his vision, the better. The more elements he can grasp and hold in mind at once, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappears. We see better. We also see further.≫ (Fisher 1892, p. 119)

These words are quite optimistic but quite distant from Walras' heartfelt expectation that in the future, supply and demand, or general equilibrium theory, will be put next to Newton and Keplero's discoveries of the laws of celestial mechanics. (Walras 2014, 47 et ss.)

To sum up, Fisher seems more inclined to treat mathematics in a way more similar to modern axiomatic theory. At the same time, his attempt to show what economic equilibrium can really be like could be seen as a precursor of Herbert Scarf's computational approach to equilibrium in the seventies.(Dimand 2019, p. 35)

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